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Listing all the minimal separators of a 3-connected planar graph

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Abstract

We present an efficient algorithm that lists the minimal separators of a 3-connected planar graph in $O(n)$ per separator.

Keywords: Minimal separator; Planar graphs; Enumeration

1. Introduction

In the last ten years, minimal separators have been increasingly studied in graph theory leading to many algorithmic applications [5,9,10,12].

For example, minimal separators are an essential tool to study the treewidth and the minimum fill-in of graphs. In [5], Bodlaender et al. conjecture that for a class of graphs with a polynomial number of minimal separators, these problems can be solved in polynomial time. Bouchitté and Todinca introduced the concept of potential maximal clique [2] and showed that, if the number of potential maximal cliques is polynomial, treewidth and minimum fill-in can be solved in polynomial time. They later showed [3] that if a graph has a polynomial number of minimal separators, it has a polynomial number of potential maximal cliques. Those results rely on deep understandings of minimal separators.

Extensive research has been performed to compute the set of the minimal separators of a graph [1,6,7,11]. Berry et al. [1] proposed an algorithm of running time $O(nm)$ per separator[1] that uses the concept of generating new minimal separators from a previous minimal separator S by finding the minimal separators contained in $S \cup N(x)$ for $x \in S$. This simple process can generate all the minimal separators of a graph. However, by using this algorithm a minimal separator can be generated many times.

The aim of this article is to address the problem of finding the minimal separators of a 3-connected planar graph G. In order to avoid the problem of recalculation, we define the set $\mathscr{S}_a(S, O)$ of the a, b-minimal separators S' for some b such that the connected component of a in $G \backslash S'$ contains the connected component of a in $G \backslash S$ but avoids the set O. Therefore, it is possible to ensure that a given minimal separator is never computed more than five times.

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[1] The authors only proved a running time of $O(n^3)$ but the actual bound is $O(nm)$ [8].

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2. Definitions

Throughout this paper, $G=(V,E)$ is a 3-connected graph without loops with $n=|V|$ and $m=|E|$. For $x\in V$, $N(x)=\{y|(x,y)\in E\}$ and for $C\subseteq V$, $N(C)=\{y\notin C|\exists x\in C,(x,y)\in E\}$. When the sets A and B are disjoint, their union is denoted by $A\sqcup B$.

A set $S\subseteq V$ is a *separator* if $G\backslash S$ has at least two connected components, an a,b-*separator* if a and b are in different connected components of $G\backslash S$, an a,b-*minimal separator* if no proper subset of S is an a,b-separator. The connected component of a in $G\backslash S$ is $C_a(S)$. The component $C_a(S)$ *is a full connected component* if $N(C_a(S))=S$. For an a,b-minimal separator S, both $C_a(S)$ and $C_b(S)$ are full. A set S is a *minimal separator* if there exist a and b such that S is an a,b-minimal separator or, which is equivalent, if it has at least two full connected components. An $a,*$-*minimal separator* of a graph $G=(V,E)$ is an a,b-minimal separator of G for some $b\in V$. The set of the $a,*$-minimal separators is denoted by $\mathscr{S}_a$ and the set of the minimal separators of G is denoted by $\mathscr{S}(G)$.

It is possible to order the $a,*$-minimal separators in the following way:

$$S_1 \preccurlyeq_a S_2 \text{ if } C_a(S_1)\subseteq C_a(S_2).$$

The minimal separator S_1 is *closer* to a than S_2. The set of a,b-minimal separators is a lattice for the relation $\preccurlyeq_a$ [4] but we only need the following weaker lemma:

Lemma 1. *Let C be a set of vertices of a graph G inducing a connected subgraph of G, a be a vertex of C and b be a vertex of $G\backslash(C\cup N(C))$.*

The neighbour S of $C_b(C\cup N(C))$ is an a,b-minimal separator such that C is a subset of $C_a(S)$ that is closer to a than any a,b-minimal separator S' such that C is a subset of $C_a(S')$.

Proof. By construction, C is a subset of $C_a(S)$. By definition, the component $C_b(S)$ is full and since S is a subset of $N(C)$, the component $C_a(S)$ is also a full component which implies that S is an a,b-minimal separator.

Let S' be an a,b-minimal separator such that C is a subset of $C_a(S')$. Let p be a path in $C_b(S')$ with b as one of its ends. The vertices of S' are at least at distance 1 of C so the vertices of p are at least at distance 2 of C. Since S is a subset of $N(C)$, $p\cap S=\emptyset$. In other words p is a subset of $C_b(S)$ and $C_b(S')\subseteq C_b(S)$. This last inclusion implies that $C_a(S)\subseteq C_a(S')$ i.e. S is closer to a than S'. □

For S an $a,*$-minimal separator and $O\subseteq V$, the set $\mathscr{S}_a(S,O)$ is *the set of the $a,*$-minimal separators S' further from a than S and such that $O\cap C_a(S')=\emptyset$*. If $x\in V$, the set $\mathscr{S}_a^x(S,O)$ is *the set of $S'\in\mathscr{S}_a(S,O)$ such that $x\in C_a(S')$*.

Remark 2. If $x\in S$, then $\mathscr{S}_a(S,O)$ is the disjoint union

$$\mathscr{S}_a(S,O\cup\{x\})\sqcup\mathscr{S}_a^x(S,O).$$

More precisely, if $(S_i)_{i\in I}$ are the elements of $\mathscr{S}_a^x(S,O)$ closest to a, then

$$\mathscr{S}_a(S,O)=\mathscr{S}_a(S,O\cup\{x\})\sqcup\left(\bigcup_{i\in I}\mathscr{S}_a(S_i,O)\right).$$

This gives the skeleton of an algorithm to compute the set $\mathscr{S}_a(S,O)$.

Remark 3. If S belongs to $\mathscr{S}_a^x(S,O)$, then $\mathscr{S}_a^x(S,O)=\mathscr{S}_a(S,O)$.

The algorithm is based on Remarks 2 and 3. To list $\mathscr{S}_a$, the algorithm computes the sets $\mathscr{S}_a(S,\emptyset)$ for every S closest to a in $\mathscr{S}_a$. During this calculation, it computes $\mathscr{S}_a(S,O)$ with $O\subseteq S$. To do so, it chooses x in $S\backslash O$ and calculates $\mathscr{S}_a^x(S,O)$ and $\mathscr{S}_a(S,O\cup\{x\})$. The set $\mathscr{S}_a^x(S,O)$ is itself a union of $\mathscr{S}_a(S_i,O)$. But to obtain such a decomposition, one needs to find the elements of $\mathscr{S}_a^x(S,O)$ closest to a, which the following proposition does.