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On the orthogonal Latin squares polytope[☆]

G. Appa[a], D. Magos[b,*], I. Mourtos[c,1], J.C.M. Janssen[d]

[a] Operational Research Department, London School of Economics and Political Science, Houghton Street, London WC2A2AE, UK
[b] Department of Informatics, Technological Educational Institute (T.E.I.) of Athens, Ag. Spyridonos Str., 12210 Egaleo, Athens, Greece
[c] Department of Economics, University of Patras, Rio 26500, Patras, Greece
[d] Department of Mathematics and Statistics, Dalhousie University, Halifax NS, Canada B3H 3J5

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Abstract

Since 1782, when Euler addressed the question of existence of a pair of orthogonal Latin squares (OLS) by stating his famous conjecture, these structures have remained an active area of research. In this paper, we examine the polyhedral aspects of OLS. In particular, we establish the dimension of the OLS polytope, describe *all* cliques of the underlying intersection graph and categorize them into three classes. Two of these classes are shown to induce facet-defining inequalities of Chvátal rank two. For each such class, we provide a polynomial separation algorithm of the lowest possible complexity.

Keywords: Orthogonal Latin squares; Polyhedral combinatorics; Clique facets

1. Introduction and motivation

A Latin square $\mathbb{L}$ of order n is an $n \times n$ matrix on n symbols, each occurring exactly once in every row and column. Without loss of generality, we assume the n symbols to be the integers $1, 2, \ldots, n$. Two Latin squares $\mathbb{L}_1 = \|a_{ij}\|$, $\mathbb{L}_2 = \|b_{ij}\|$ are called *orthogonal* if every ordered pair of symbols occurs exactly once among the n^2 pairs (a_{ij}, b_{ij}) : $i, j = 1, 2, \ldots, n$. An example of a pair of orthogonal Latin squares (OLS) of order 4 is illustrated in Table 1 (an OLS configuration will be illustrated as an $n \times n$ array containing in row i, column j the pair (a_{ij}, b_{ij})).

The OLS definition is naturally extended to a set of more than two Latin squares, which are said to be *mutually* orthogonal (MOLS) if and only if they are pairwise orthogonal. MOLS were introduced by L. Euler through the 36-officers problem, which asks for a pair of OLS of order 6. Having failed to find such a configuration, Euler conjectured that there exists no pair of OLS of order $n = 2 \bmod 4$ [9,14]. The infeasibility for $n = 6$ was first proven in [20]. However, it was the falsity of this conjecture, for $n > 6$, that revived the interest in MOLS [7]. At present, MOLS remain an active area of research because of their theoretical properties and their applications in a variety of fields (see [9,10,14]).

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[*] Corresponding author. 30 Theodorou Geometrou Str., 11743 Athens, Greece.
E-mail addresses: g.appa@lse.ac.uk (G. Appa), dmagos@teiath.gr (D. Magos), imourtos@upatras.gr (I. Mourtos), janssen@mathstat.dal.ca (J.C.M. Janssen).
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Table 1
An OLS configuration of order 4

(1,1)	(2,2)	(3,3)	(4,4)
(2,3)	(1,4)	(4,1)	(3,2)
(3,4)	(4,3)	(1,2)	(2,1)
(4,2)	(3,1)	(2,4)	(1,3)

In spite of the sizeable literature, the facial structure of the polytopes associated with MOLS has not been studied. This work initiates a study on the subject by examining the OLS polytope. In particular, we establish the dimension of the OLS polytope, identify *all* classes of facet-defining inequalities arising from cliques of the associated intersection graph and show how to separate them in polynomial time. These results are directly applicable, within the context of a *Branch and Cut* (BC) scheme, to the problem of minimizing (maximizing) a linear function over the OLS polytope (see the 4-*index planar assignment* problem discussed in the following section). Such schemes, integrating the results introduced here with logic-based methods, have been implemented for constructing OLS pairs algorithmically (see [4]). Variants of these implementations can be used to check whether a particular Latin square has an orthogonal mate (see [1] for models related to this problem), whether a partially filled OLS pair can be completed, etc.

Furthermore, this work can be considered as a first step towards the study of polytopes related to open theoretical problems on MOLS. One such problem concerns the existence of a set of three MOLS of order 10. A more general question addresses the issue of the existence of $n-1$ MOLS of order n, for n not being a prime power. This is equivalent to asking for a projective plane of a non-prime-power order, a fundamental question in finite geometry. In general, the polyhedral analysis of MOLS polytopes could help to answer questions rising in various fields of combinatorics like affine designs, (t, m, s)-nets, graph factorizations (see [14] for an extensive discussion), etc.

Focusing on the OLS polytope, we show (in the next section) that it is related to the polytopes examined in [6,12]. However, it presents irregularities not encountered in previous studies, e.g. it is empty for $n = 2, 6$. For this reason, the OLS structure appears interesting from the perspective of polyhedral combinatorics.

The rest of this paper is organized as follows. Section 2 presents the mathematical formulation of the problem and discusses related problems. The associated intersection graph and all its cliques are described in Section 3. In Section 4, we establish the dimension of the underlying polytope and show that two classes of clique inequalities induce facets of this polytope. We also prove that these inequalities are of Chvátal rank 2. Separation algorithms for these two families of inequalities are presented in Section 5.

2. The OLS polytope and related structures

Several different mathematical programming formulations for the OLS problem are given in [3]. The integer programming (IP) model reproduced next is attributed to D. Gale (in [8]). Let I, J, K, L be disjoint n-sets, indexing the rows, columns and symbols of the two Latin squares, respectively. Let also the binary variable x_{ijkl} be 1, if pair (k, l) appears in cell (i, j), and 0, otherwise. The constraints of the model are

$$\sum\{x_{ijkl}\colon\ i \in I, j \in J\} = 1, \quad \forall k \in K, \ \ l \in L, \tag{2.1}$$

$$\sum\{x_{ijkl}\colon\ j \in J, k \in K\} = 1, \quad \forall i \in I, \ \ l \in L, \tag{2.2}$$

$$\sum\{x_{ijkl}\colon\ i \in I, k \in K\} = 1, \quad \forall j \in J, \ \ l \in L, \tag{2.3}$$

$$\sum\{x_{ijkl}\colon\ k \in K, l \in L\} = 1, \quad \forall i \in I, \ \ j \in J, \tag{2.4}$$

$$\sum\{x_{ijkl}\colon\ i \in I, l \in L\} = 1, \quad \forall j \in J, \ \ k \in K, \tag{2.5}$$

$$\sum\{x_{ijkl}\colon\ j \in J, l \in L\} = 1, \quad \forall i \in I, \ \ k \in K, \tag{2.6}$$

$$x_{ijkl} \in \{0, 1\}, \quad \forall i \in I, \ \ j \in J, \ \ k \in K, \ \ l \in L. \tag{2.7}$$