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Total Coloring of Certain Classes of Product Graphs

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Abstract

A total coloring of a graph is an assignment of colors to all the elements of the graph such that no two adjacent or incident elements receive the same color. In this paper, we prove the tight bound of Behzad and Vizing conjecture on total coloring for Compound graph of G and H, where G and H are any graphs.

Keywords: Total coloring, Compound graph, Product graph, Bipartite graphs, Complete graphs.

1 Introduction

All graphs considered here are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph with the sets of vertices and edges $V(G)$ and $E(G)$, respectively. A *total coloring* of G is a mapping $f : V(G) \cup E(G) \to C$, where C is a set of colors, satisfying the following three conditions (a)-(c):

(a) $f(u) \neq f(v)$ for any two adjacent vertices $u, v \in V(G)$

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(b) $f(e) \neq f(e')$ for any two adjacent edges $e, e' \in E(G)$ and

(c) $f(v) \neq f(e)$ for any vertex $v \in V(G)$ and any edge $e \in E(G)$ incident to v.

The *total chromatic number* of a graph G, denoted by $\chi''(G)$, is the minimum number of colors that suffice in a total coloring. It is clear that $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. Behzad [1] and Vizing [24] conjectured (Total Coloring Conjecture (TCC)) that for every graph G, $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$. If a graph G is total colorable with $\Delta(G) + 1$ colors then the graph is called type - I, and if it is total colorable with $\Delta(G)+2$ colors, then it is type - II. This conjecture was verified by Rosenfeld [21] and Vijayaditya [23] for $\Delta(G) = 3$ and by Kostochka [15,16,17] for $\Delta(G) \leq 5$. For planar graphs, the conjecture was verified by Borodin [3] for $\Delta(G) \geq 9$. In 1992, Yap and Chew [25] proved that any graph G has a total coloring with at most $\Delta(G) + 2$ colors if $\Delta(G) \geq |V(G)| - 5$, where $|V(G)|$ is the number of vertices in G. In 1993, Hilton and Hind [13] proved that any graph G has a total coloring with at most $\Delta(G) + 2$ colors if $\Delta(G) \geq \frac{3}{4}|V(G)|$. It is known that the total coloring problem, which asks to find a total coloring of a given graph G with the minimum number of colors, is NP-hard [3]. In particular, McDiarmid [5] and Arroyo [22] proved that the problem of determining the total coloring of μ-regular bipartite graph is NP-hard, $\mu \geq 3$.

The *Compound graph* of a graph G by a graph H is a graph obtained by the following way: we replace the vertices of G by copies of H, and we add an edges to some of vertices of two of these copies iff the corresponding vertices of G are adjacent. The compound graph of $C_4[K_5]$ shown in Fig. 1.

In the following, we will denote by $G[H]$ any compound of a graph G by a graph H, where there exactly one perfect matching between two copies of H iff the corresponding vertices of G are adjacent. This operation clearly does not define a unique graph; however, in the following we will write $G' = G_1[G_2]$ if G' is compound graph of the form $G_1[G_2]$. This graph was introduced by Bermond, Delorme and Quisquater [2] in 1970. The following theorems are due to Yap [25].

Theorem 1.1 *Let K_n be the complete graph, then*

$$\chi''(K_n) = \begin{cases} n, \text{ if } n \text{ is odd} \\ n+1, \text{ if } n \text{ is even.} \end{cases}$$

Theorem 1.2 *Let C_n be the cycle graph, then* $\chi''(C_n) = \begin{cases} 3, \text{ if } n \equiv 0 \mod 3 \\ 4, \text{ otherwise.} \end{cases}$