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Computational Solution of an Old Tower of Hanoi Problem ⋆

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Abstract

This is the amazing story of an innocent looking mathematical puzzle turning into a serious research topic in graph theory, integer sequences, and algorithms. The *Tower of Hanoi* and *The Reve's Puzzle* of Lucas and Dudeney, respectively, induced a wealth of interesting mathematical and algorithmic challenges over more than a century. Although some part of the most intriguing question, the *Frame-Stewart Conjecture*, has recently been solved, several of the original tasks posed by Dudeney remained intractable. We present the history and theory of these questions and a computational approach which allowed us to solve a 104 years old problem of Dudeney, namely the proof of minimality of an algorithm producing paths between perfect states of the Tower of Hanoi with 5 pegs and 20 discs. Many questions about the metric properties of *Hanoi graphs* remain open, however, and have to be treated by analytical and computational methods in the future.

Keywords: Tower of Hanoi, graph distance, integer sequences, breadth-first search

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1 Lucas and Dudeney

In 1883, the French number theorist Édouard Lucas brought out an innocent looking mathematical puzzle that he called *The Tower of Hanoi* (ToH) [1] and which was to become maybe the most prominent example of how recreational mathematics can influence such serious topics as the study of integer sequences or graph theory. See the book [20] for a comprehensive overview of the development until 2012. With his four-volume work [2] [i] Lucas was one of the poineers of popularizing mathematics while keeping a high standard of rigor.

In the tradition of Lucas, Henry Ernest Dudeney published mathematical puzzles in English magazines starting in the 1890s. Without mentioning his French antetype, [ii] Dudeney presented *The Tower of Bramah* in 1896 in [4] under the pen name "Sphinx", thereby citing Ball's famous description (cf. [20, p. 1f]) almost verbally, even with the strange spelling of the Hindu God of Creation. Dudeney posed the usual questions for the length of a minimal solution for the ToH with three pegs and 8 or 64 discs. For the first time he also asked for the best first move. His answers contain a clerical error (225 moves for 8 discs instead of 255) and no proofs.

Only three months later and again in *The Weekly Dispatch*, Dudeney, or rather "Sphinx Junior", came up with the challenge to solve the problem in the presence of four pegs (or pins) with 10 discs [5]. So the *terminus ante quem* for the ToH with four pegs is 1896–11–15. In the solution, Dudeney claims two weeks later that "[t]he feat can be performed in fourty-nine moves, which is the correct answer." Just like Fermat he denounces the lack of space for not going "fully into the subject", but reveals "the Junior's method of working the puzzle", namely to make successive piles of 6 (1 to 6) and 3 (7 to 9) discs, then removing disc 10 taking 17, 7, and 1 moves, respectively. Finally, the two piles are replaced on disc 10 in inverse order, so that in total 49 moves are performed, "the fewest posssible". Again, there is no argument given for the latter claim. It would take more than a hundred years before the claim was eventually proved in [17]! But the stage was set for what would later become known as the *Frame-Stewart conjecture*, and *Frame's algorithm* was anticipated.

In *The London Magazine* of 1902 Dudeney posed the problem to find a shortest solution for the ToH with four pegs again; see [6, p. 367f]. He does

[i] Two more volumes were planned, but not achieved due to Lucas's untimely death; cf. [3, p. 210].
[ii] Only in 1908, Dudeney would mention the name of the inventor in [10, p. 784].