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Many disjoint edges in topological graphs

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Abstract

A *monotone cylindrical* graph is a topological graph drawn on an open cylinder with an infinite vertical axis satisfying the condition that every vertical line intersects every edge at most once. It is called *simple* if any pair of its edges have at most one point in common: an endpoint or a point at which they properly cross. We say that two edges are *disjoint* if they do not intersect. We show that every simple complete monotone cylindrical graph on n vertices contains $\Omega(n^{1-\epsilon})$ pairwise disjoint edges for any $\epsilon > 0$. As a consequence, we show that every simple complete topological graph (drawn in the plane) with n vertices contains $\Omega(n^{\frac{1}{2}-\epsilon})$ pairwise disjoint edges for any $\epsilon > 0$. This improves the previous lower bound of $\Omega(n^{\frac{1}{3}})$ by Suk which was reproved by Fulek and Ruiz-Vargas. We remark that our proof implies a polynomial time algorithm for finding this set of pairwise disjoint edges.

Keywords: Topological graphs, disjoint edges, cylindrical graphs, complete graph, graph drawings.

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1 Introduction

A *topological graph* is a graph drawn on the plane so that its vertices are represented by points and its edges are represented by Jordan arcs connecting the respective endpoints. Moreover, in topological graphs we do not allow overlapping edges or edges passing through a vertex. A *topological graph* is *simple* if every pair of its edges meet at most once, either in a common vertex or at a proper crossing. We use the words "vertex" and "edge" in both contexts, when referring to the elements of an abstract graph and also when referring to their corresponding drawings. A graph is *complete* if there is an edge between every pair of vertices. We say that two edges are *disjoint* if they do not intersect. Throughout this note n denotes the number of vertices in a graph.

By applying a theorem of Erdős and Hajnal [5], every complete n-vertex simple topological graph contains $e^{\Omega(\sqrt{\log n})}$ edges that are either pairwise disjoint or pairwise crossing. However, it was thought [15] that this bound is far from optimal. Fox and Pach [8] showed that there exists a constant $\delta > 0$ such that every complete n-vertex simple topological graph contains $\Omega(n^{\delta})$ pairwise crossing edges. In 2003, Pach, Solymosi, and Tóth [15] showed that every complete n-vertex simple topological graph has at least $\Omega(\log^{1/8} n)$ pairwise disjoint edges. This lower bound was later improved by Pach and Tóth [16] to $\Omega(\log n/ \log\log n)$. Fox and Sudakov [10] improved this to $\Omega(\log^{1+\epsilon} n)$, where ϵ is a very small constant. Furthermore, the previous two bounds hold for dense simple topological graphs. Pach and Tóth conjectured (see Problem 5, Chapter 9.5 in [3]) that there exists a constant $\delta > 0$ such that every complete n-vertex simple topological graph has at least $\Omega(n^{\delta})$ pairwise disjoint edges. Using the existence of a perfect matching with a low stabbing number for set systems with polynomially bounded dual shattered function [4], Suk [17] settled this conjecture by showing that there are always at least $\Omega(n^{1/3})$ pairwise disjoint edges. This was later reproved by Fulek and Ruiz-Vargas [12] using completely different techniques. We are now able to improve the lower bound to $\Omega(n^{1/2-\epsilon})$ for any $\epsilon > 0$.

Theorem 1.1 *A complete simple topological graph on n vertices, which is drawn on the plane, contains $\Omega(n^{\frac{1}{2}-\epsilon})$ pairwise disjoint edges.*

To the best of our knowledge, no sub-linear upper bound is known for this problem.