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Diagonalization of generalized lollipop graphs [1]

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Abstract

Let G be a graph formed by connecting a tree and a threshold graph with an edge between their respective roots. Let A be the adjacency matrix of G and $x \in \mathbb{R}$. We give an $O(n)$ algorithm for constructing a diagonal matrix D congruent to $A + xI_n$, allowing us to locate eigenvalues of A and obtain spectral results.

Keywords: adjacency matrix, diagonalization, eigenvalue, threshold graph, tree.

1 Introduction

Given a class $\mathcal{C}$ of real symmetric matrices, it is important to efficiently locate the eigenvalues of any $A \in \mathcal{C}$. By locate we mean to compute the number of eigenvalues in any given real interval. One way to accomplishing this is to construct a diagonal matrix D congruent to $A + xI_n$. By Sylvester's Law of Inertia, one can show that the number of positive (negative) entries in D is the number of eigenvalues of A greater than (less than) $-x$. This algorithm could be useful in numeric divide-and-conquer computations to determine the

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spectrum of A. More importantly, however, such an algorithm can be used as an analytical tool to obtain general properties of the matrices in $\mathcal{C}$. This has been done for the class of adjacency matrices of trees and for the class of adjacency matrices of threshold graphs as described below.

In [7] an algorithm was given for computing a diagonal matrix D congruent to $A+xI_n$, for A the adjacency matrix of a tree T, and $x \in \mathbb{R}$. The bottom-up algorithm operates directly on the tree; values are stored in the nodes, and at termination, the final values are the diagonal entries of D. This diagonalization approach has proved to be a powerful tool in obtaining new information about the spectral properties of trees. The algorithm and its variations have been used to show at least half the Laplacian eigenvalues of a tree with n vertices are smaller than two [2], and to order trees with diameter three by their Laplacian energy [3]. Using diagonalization, in [11] infinite families of trees with integral index were obtained. Using the localization algorithm, in [4] it was shown that the n-vertex star has the highest Laplacian energy over all n-vertex trees, and the bound $(n-1)+2k-1-\frac{2k-2}{n}$ was obtained for the sum S_k of the k largest Laplacian eigenvalues for any tree T. Using this upper bound, Brouwer's conjecture that $S_k \leq m + \binom{k+1}{2}$ for graphs with m edges, was settled for unicyclic and bicyclic graphs [12].

Threshold graphs are an important class of graphs because of their numerous applications [10]. We construct threshold graphs through an iterative process which starts with an isolated vertex we call the *root*. At each step, either a new isolated vertex is added, or a (dominating) vertex adjacent to all other vertices is added. We represent a threshold graph G on n vertices using a binary string $b_1 \dots b_n$, called a *creation sequence*. Here $b_i = 0$ if vertex v_i was added as an isolated vertex, and $b_i = 1$ if v_i was added as a dominating vertex. We take b_1 to be zero. In constructing an adjacency matrix, we order the vertices in the same way they are given in their creation sequence.

In [8] an algorithm was given for computing a diagonal matrix D congruent to $A+xI_n$, for A the adjacency matrix of a threshold graph G and $x \in \mathbb{R}$. Like its counterpart for trees, the algorithm utilized only storage for the diagonal elements as well as the creation sequence of the graph. This algorithm has been used to obtain interesting theorems about the spectrum of the adjacency matrix of threshold graphs. For example, all eigenvalues $\lambda \neq -1, 0$ are simple and no eigenvalue belongs to the interval $(-1, 0)$ (see [8,9]).

Recall that a *lollipop* [1] is sometimes defined to be a path P_j connected to the complete graph K_k. We define a *generalized lollipop* to be a graph formed by connecting the root of a tree and the root of a threshold graph with an edge. Our goal is to give an algorithm for constructing a diagonal matrix congruent