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Pebbling in 2-Paths

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Abstract

Graph pebbling is a network model for transporting discrete resources that are consumed in transit. Deciding whether a given configuration on a particular graph can reach a specified target is NP-complete, even for diameter two graphs, and deciding whether the pebbling number has a prescribed upper bound is Π_2^{P}-complete. Recently we proved that the pebbling number of a split graph can be computed in polynomial time. This paper continues the program of finding other polynomial classes, moving away from the large tree width, small diameter case (such as split graphs) to small tree width, large diameter, beginning an investigation on the important subfamily of chordal graphs called k-trees. In particular, we provide a formula for the pebbling number of any 2-path.

Keywords: pebbling number, k-trees, k-paths, Class 0, complexity

1 Introduction

The fundamental question in graph pebbling is whether a given supply (*configuration*) of discrete pebbles on the vertices of a connected graph can satisfy a

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particular set of demands on the vertices. The operation of pebble movement across an edge is called a *pebbling step*: while two pebbles cross the edge, only one arrives at the opposite end, as the other is consumed. The most studied scenario involves the demand of one pebble on a single *root* vertex r. Satisfying this demand is often referred to as *reaching* or *solving* r, and configurations are consequently called either r-*solvable* or r-*unsolvable*.

The *size* $|C|$ of a configuration $C : V \to \mathbb{N} = \{0, 1, \ldots\}$ is its total number of pebbles $\sum_{v \in V} C(v)$. The *pebbling number* $\pi(G) = \max_{r \in V} \pi(G, r)$, where $\pi(G, r)$ is defined to be the minimum number s so that every configuration of size at least s is r-solvable. Simple sharp lower bounds like $\pi(G) \geq n$ and $\pi(G) \geq 2^{\mathsf{diam}(G)}$ are easily derived. Graphs satisfying $\pi(G) = n$ are called *Class* 0 and are a topic of much interest. Recent chapters in [6] and [7] include variations on the theme such as k-pebbling, fractional pebbling, optimal pebbling, cover pebbling, and pebbling thresholds, as well as applications to combinatorial number theory, combinatorial group theory, and p-adic diophantine equations, and also contain important open problems in the field.

Computing graph the pebbling number is difficult in general. The problem of deciding if a given configuration on a graph can reach a particular vertex was shown in [8] and [10] to be NP-complete, even for diameter two graphs ([4]) or planar graphs ([9]). Interestingly, the problem was shown in [9] to be in P for graphs that are both planar and diameter two, as well as for outerplanar graphs (which include 2-trees). The problem of deciding whether a graph G has pebbling number at most k was shown in [10] to be Π_2^{P}-complete.

In contrast, the pebbling number is known for many graphs. For example, in [11] the pebbling number of a diameter 2 graph G was determined to be n or $n+1$. Moreover, [3] and [2] characterized those graphs having $\pi(G) = n+1$, and it was shown in [5] that one can recognize such graphs in quartic time. Beginning a program to study for which graphs their pebbling number can be computed in polynomial time, the authors of [1] produced a formula for the family of split graphs that involves several cases. For a given graph, finding to which case it belongs takes $O(n^{1.41})$ time. The authors also conjectured that the pebbling number of a chordal graph of bounded diameter can be computed in polynomial time.

In contrast to the small diameter, large tree width case of split graphs, we turn here to chordal graphs with large diameter and small tree width. In this paper we study 2-paths, the sub-class of 2-trees whose graphs have exactly two simplicial vertices, and prove an exact formula that can be computed in linear time.