Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in DISCRETE MATHEMATICS
Electronic Notes in Discrete Mathematics 50 (2015) 175–180
www.elsevier.com/locate/endm

On containment graphs of paths in a tree

Liliana Alcón [a] Noemí Gudiño [a,b] Marisa Gutierrez [a,b]

[a] *Dto. de Matemática, FCE-UNLP, La Plata, Argentina*
[b] *CONICET*

Abstract

The aim of the present work is the study of those posets that admit a containment model mapping vertices into paths of a tree, and their comparability graphs named CPT graphs. Answering a question posed by J.Spinrad, we prove that the dimension of CPT graphs is unbounded. We show that every tree is CPT. When every transitive orientation of the edges of a comparability graph G defines a CPT poset, we say that G is strong-CPT. When both some transitive orientation of G and its reverse are CPT, we say that G is dually-CPT. Split comparability graphs that are dually-CPT or strong-CPT are characterized by forbidden induced subgraphs.

Keywords: posets, comparability graphs, permutation graphs, split graphs.

1 Introduction and previous results

A *partially ordered set* or *poset* is a pair $\mathbf{P} = (X, P)$ where X is a finite non-empty set, whose elements are called *vertices*, and P is a reflexive, anti-symmetric and transitive binary relation on X. As usual, we write $x \leq y$ in $\mathbf{P}$ for $(x, y) \in P$; and $x < y$ in $\mathbf{P}$ when $(x, y) \in P$ and $x \neq y$. If $x < y$ or $y < x$, we say that x and y are *comparable* in $\mathbf{P}$ and write $x \perp y$. The set $\{x \in X : x < z\}$ is denoted by $D(z)$, we let $D[z] = D(z) \cup \{z\}$. The *restriction* of the relation P to a subset Y of X is denoted by $P(Y)$. We call $\mathbf{P}(Y)$ to the subposet $(Y, P(Y))$ of $\mathbf{P}$.

http://dx.doi.org/10.1016/j.endm.2015.07.030

A *containment model* $M_{\mathbf{P}}$ of a poset $\mathbf{P} = (X, P)$ maps each element x of X into a set M_x in such a way that $x < y$ in $\mathbf{P}$ iff M_x is a proper subset of M_y, i.e.

$$x < y \text{ in } P \Leftrightarrow M_x \subset M_y.$$

We identify the containment model $M_{\mathbf{P}}$ with the set family $(M_x)_{x \in X}$. Notice that a containment model can always be obtained by mapping each vertex x into the set $D[x]$.

Many classes of posets, grouped together under the generic name of *geometric containment orders*, have been defined by imposing geometric conditions to the sets in which the elements of the poset are mapped: they must be intervals of the line, angular regions in the plane, d-boxes in the d-Euclidean space, d-spheres in the d-Euclidean space [4,9,10]. In [4], it is proved that each poset admits a containment model using subtrees of a star (a tree with a unique vertex with degree greater than one).

In this paper, we initiate the study of those posets that admit a containment model mapping vertices into paths of a tree, and their comparability graphs. Spinrad in [7] and Golumbic in [5] wonder about the properties of these graphs. All the results contained in the present work without a reference to their authors are novel, their proofs are omitted due to space limitations.

The comparability graph $G_{\mathbf{P}}$ of a poset $\mathbf{P} = (X, P)$ is the simple graph with vertex set $V(G_{\mathbf{P}}) = X$ and edge set $E(G_{\mathbf{P}}) = \{xy : x \perp y\}$. A graph G is a *comparability graph* if there exists some poset $\mathbf{P}$ such that $G = G_{\mathbf{P}}$. Comparability graphs were characterized by forbidden induced subgraph by Gallai. A *transitive orientation* $\overrightarrow{E}$ of a graph $G = (V, E)$ is an assignment of one of the two possible directions, $\overrightarrow{xy}$ or $\overrightarrow{yx}$, to each edge $xy \in E$ in such a way that if $\overrightarrow{xy} \in \overrightarrow{E}$ and $\overrightarrow{yz} \in \overrightarrow{E}$ then $\overrightarrow{xz} \in \overrightarrow{E}$. Ghouila-Houri proved that the graphs whose edges can be transitively oriented are exactly comparability graphs. For further information in this topic, see [1,3,9].

Dushnik and Miller defined the *dimension* of a poset $\mathbf{P}$, denoted by $dim(\mathbf{P})$, as the minimum number of linear orders whose intersection is $\mathbf{P}$ [2]. The dimension of a poset is an invariant of the associated comparability graph [9], this means that if $\mathbf{P}$ and $\mathbf{P'}$ are posets satisfying $G_{\mathbf{P}} = G_{\mathbf{P'}}$ then $dim(\mathbf{P}) = dim(\mathbf{P'})$. The dimension of a comparability graph G is defined as the dimension of any poset $\mathbf{P}$ such that $G = G_{\mathbf{P}}$.

The *dual* of a poset $\mathbf{P} = (X, P)$ is the poset $\mathbf{P}^d = (X, P^d)$ where $x < y$ in $\mathbf{P}^d$ iff $y < x$ in $\mathbf{P}$. Notice that $G_{\mathbf{P}} = G_{\mathbf{P}^d}$ and, obviously, $dim(\mathbf{P}) = dim(\mathbf{P}^d)$.

In [2], it was proved that $dim(\mathbf{P}) \leq 2$ iff $\mathbf{P}$ admits a containment model mapping vertices into intervals of the line. Therefore posets with dimension at most 2 also appear in the literature as *containment orders of intervals*, we