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Edge-colorings avoiding fixed rainbow stars

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Abstract

We consider an extremal problem motivated by a question of Erdős and Rothschild and by a paper of Balogh, who considered edge-colorings of graphs avoiding fixed subgraphs with a prescribed coloring. Given $r \geq t \geq 2$, we look for n-vertex graphs that admit the maximum number of r-edge-colorings such that at most $t-1$ colors appear in edges incident with each vertex. For large n, we show that, with the exception of the case $t = 2$, the complete graph K_n is always the unique extremal graph. We also consider generalizations of this problem.

Keywords: Extremal Graph Theory, restricted edge-colorings, rainbow colorings.

1 Introduction and main results

We consider edge-colorings of graphs that satisfy a certain property. Given a number r of colors and a graph F, an r-*pattern* P of F is a partition of its edge set into r (possibly empty) classes. An edge-coloring (not necessarily proper) of a host graph H is said to be (F, P)-*free* if H does not contain a copy of F in which the partition of the edge set induced by the coloring is isomorphic to P. If at most r colors are used, we call it an (F, P)-free r-coloring of H. For example, if the pattern of F consists of a single class, no monochromatic

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copy of F should arise in H. We ask for the n-vertex host graphs H (among all n-vertex graphs) which allow the largest number of (F, P)-free r-colorings.

Questions of this type have been first considered by Erdős and Rothschild [5], who asked whether considering edge-colorings avoiding a monochromatic copy of F would lead to extremal configurations that are different from those of the Turán problem. Indeed, F-free graphs on n-vertices are natural candidates for a large number of colorings, since any r-coloring of their edge set obviously does not produce a monochromatic copy of F (or a copy of F with any given pattern), so that (Turán) F-extremal graphs admit $r^{\mathrm{ex}(n,F)}$ such colorings, where, as usual, $\mathrm{ex}(n, F)$ is the maximum number of edges in an n-vertex F-free graph. Erdős and Rothschild [5] conjectured that, for every $\ell \geq 3$ and $n > n_0(\ell)$, any n-vertex graph with the largest number of K_ℓ-free 2-colorings is isomorphic to the $(\ell - 1)$-partite Turán graph, which was proven for $\ell = 3$ by Yuster [9] and for $\ell \geq 4$ by Alon, Balogh, Keevash, and Sudakov [1], who also showed that the same conclusion holds in the case $r = 3$. However, for $r \geq 4$ colors, the Turán graph for K_ℓ is no longer optimal, and the situation becomes more complicated; in fact, extremal configurations are not known unless $r = 4$ and $F \in \{K_3, K_4\}$, see Pikhurko and Yilma [8].

Balogh [2] was the first to consider colorings avoiding fixed patterns that are not monochromatic. More precisely, he showed that the $(\ell - 1)$-partite Turán graph is still optimal for $r = 2$ colors when forbidding any 2-pattern of K_ℓ. However, this does not hold in general for $r = 3$ colors and arbitrary 3-patterns of K_ℓ. Indeed, consider $F = K_3$ and let P be a partition of K_3 into three classes containing one edge each, so that we are looking for 3-colorings with no *rainbow* triangle. If we color the complete graph K_n with any two of the three colors available, there is no rainbow copy of K_3, which gives at least $3 \cdot 2^{\binom{n}{2}} - 3$ distinct (K_3, P)-free colorings of K_n, and is more than $3^{\mathrm{ex}(n,K_3)} = 3^{n^2/4+O(1)}$. This suggests that the study of colorings that avoid general patterns, and in particular rainbow patterns, deserves more attention. In connection with this, it was recently proven with the Regularity Lemma that, for n sufficiently large, K_n is indeed optimal for rainbow triangles [4].

Stars have played an important rôle in these developments. As it turns out, monochromatic stars $F = S_t$ with $t \geq 3$ edges were the first instances for which it was shown [7] that, for any fixed $r \geq 2$, F-extremal graphs (in this case, $(t - 1)$-regular graphs for n even) do not admit the largest number of r-colorings with no monochromatic copy of F. However, extremal n-vertex graphs for forbidden monochromatic S_t are not known for any $r \geq 2$ and $t \geq 3$.

In this paper, our motivation was to study r-colorings that avoid rainbow-colored stars S_t, that is, we let $F = S_t$ and we consider the pattern where