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On the average path length of a cycle plus random edges

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Abstract

The average path length of a connected undirected graph is the expected distance of a randomly chosen pair of vertices. The n-cycle initially has an average path length of about $n/4$; it is known that an additional ϵn random edges is enough to reduce it to of order at most $\mathcal{O}(\epsilon^{-1}\log n)$, with high probability. We give more precise upper and lower bounds for the average path length when ϵn random edges are added to the n-cycle.

Keywords: average path length; martingales; random graph; random process; small-world network.

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1 Introduction

For an undirected graph G of order n, we denote by $d(u,v)$ the distance between two vertices u and v, which is the length of a shortest path connecting u and v. The diameter and the average path length of G are respectively defined as follows:

$$D(G) := \max_{(u,v)\in V(G)^2} d(u,v) \text{ and } L(G) := \frac{1}{n(n-1)} \sum_{(u,v)\in V(G)^2} d(u,v),$$

where $V(G)$ is the set of vertices of G. We shall only work with connected graphs, so the above quantities always exist.

Graphs with "small" average path length (or diameter) are of interest in many branches of mathematics but especially in the field of complex networks. Having a small average path length, is for example, an important property of the so-called small-world network, see [5] for more details. One usually generates small-world networks by perturbing regular graphs; these perturbations reduce the average path length quite significantly.

The easiest and perhaps the most natural of such perturbations is the addition of random edges. However, even in the simplest case, it is not clear how fast the average path length decreases in terms of the number of random edges introduced. In [3] for example, a random matching is added to the n-cycle to keep the regularity of the graph; the authors showed that this addition of a random matching reduces the diameter to about $\log_2 n$, with high probability.

Throughout this paper, we analyse the following model: to each non-adjacent pair of vertices in the n-cycle we add an edge with probability p, independently of the other edges. The resulting random graph is denoted by $C(n,p)$. The following is our main theorem:

Theorem 1.1 *If $(\log n)^{-a} \ll np \ll 1$ for some positive constant $a < 1$, then for any constant $\delta > 0$, the event*

$$(1-\delta)\,\frac{\log n}{\log \lambda} \le L\left(C(n,p)\right) \le (1+\delta)\,\frac{\log n}{\log \lambda} \tag{1}$$

holds with high probability, where

$$\lambda = \frac{1 + np + \sqrt{(1+np)^2 + 4np}}{2}.$$