Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in DISCRETE MATHEMATICS

Electronic Notes in Discrete Mathematics 50 (2015) 313–318 www.elsevier.com/locate/endm

Partitioning two-coloured complete multipartite graphs into monochromatic paths and cycles

Oliver Schaudt [1]

Institut für Informatik, Universität zu Köln, Cologne, Germany

Maya Stein [2,3]

Centro de Modelamiento Matemático, Universidad de Chile, Santiago, Chile

Abstract

We show that any complete k-partite graph G on n vertices, with $k \geq 3$, whose edges are two-coloured, and whose largest partition class contains at most $n/2$ vertices, can be covered with two vertex-disjoint monochromatic paths of distinct colours. This extends known results for complete and complete bipartite graphs.

Secondly, we show that in the same situation, all but $o(n)$ vertices of the graph can be covered with two vertex-disjoint monochromatic cycles of distinct colours, if colourings close to a split colouring are excluded. From this we derive that the whole graph, if large enough, may be covered with 14 vertex-disjoint monochromatic cycles.

Keywords: monochromatic path partition, monochromatic cycle partition, two-coloured graph

[1] Email: `schaudto@uni-koeln.de`

[2] Email: `mstein@dim.uchile.cl`

[3] Supported by Fondecyt Regular no. 1140766.

http://dx.doi.org/10.1016/j.endm.2015.07.052

1 Introduction

Gerencsér and Gyárfás [5] observed the vertex set of any complete graph whose edges are coloured red and blue [4] can be partitioned into a red and a blue path. This is fairly easy: just take a maximal set S of vertices that span two paths P_1, P_2, one in each colour, which only meet in one of their endvertices, call this vertex x. One quickly checks that any vertex $v \notin S$ can be used to augment S: we can add the edge xv to the path P_i of the same colour, and then go from v on reversely through P_{3-i}. It is a long-standing conjecture that this phenomenon carries over to arbitrarily many colours.

Conjecture 1.1 (Gyárfás [6]) *Let G be a complete graph whose edges are coloured with r colours. Then G can be partitioned into r monochromatic paths.*

A stronger conjecture, replacing paths by cycles, had been put forward by Erdős, Gyárfás and Pyber [4], but was recently disproved by Pokrovskiy [12] for $r \geq 3$. (Here, and throughout the paper, a cycle is allowed to consist of a single vertex or an edge, or to be totally empty.) In the case $r = 2$, however, the stronger result with cycles does hold (even with cycles of distinct colours). This used to be known as Lehel's conjecture, and was shown for all n by Bessy and Thomassé [2], after having been proved for large values of n by Łuczak, Rödl and Szemerédi [11] and by Allen [1].

Theorem 1.2 (Bessy and Thomassé [2]) *Let G be a complete graph whose edges are coloured red and blue. Then G can be partitioned into a red and a blue cycle.*

Together with Conlon, the second author showed in [3] that Theorem 1.2 literally extends to 2-local colourings: those are colourings with any number of colours, where each vertex is incident with at most two colours.

For arbitrary r, the best known bound on the number of vertex-disjoint cycles needed to cover the r-coloured complete graph K_n is $100r \log r$, if n is large, this bound is due to Gyárfás, Ruszinkó, Sárközy and Szemerédi [7]. For $r = 3$, the same authors show in [9] that there is a partition of all but $o(n)$ vertices of K_n into 3 or less monochromatic cycles. From this they deduce that 17 cycles partition the whole graph.

If one aims for similar results in complete bipartite graphs, it is reasonable to assume these are *balanced*, i.e. the two partition classes have the same size.

---

[4] Note that a colouring is never meant to be a proper colouring in this paper: any assignment of colours red any blue to the edges will do.