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Graph Saturation Games

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Abstract

We study [7,8] $\mathcal{F}$*-saturation games*, first introduced by Füredi, Reimer and Seress [3] in 1991, and named as such by West [11]. The main question is to determine the length of the game whilst avoiding various classes of graphs, playing on a large complete graph. We show lower bounds on the length of path-avoiding games, and more precise results for short paths. We show sharp results for the tree avoiding game and the star avoiding game. We also study directed analogues of these games. We show tight results on the walk-avoiding game. We also examine an intermediate game played on undirected graphs, such that there exists an orientation avoiding a given family of directed graphs, and show bounds on the score. This last game is known to be equivalent to a recent game studied in [6], and we give new bounds for biased versions of this game.

Keywords: saturation game, graph, digraph

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1 Introduction

For $\mathcal{F}$ a family of graphs, a graph G is *$\mathcal{F}$-free* if G contains no member of $\mathcal{F}$ as a subgraph, and $G \subset H$ is an *$\mathcal{F}$-saturated* subgraph of H if G is a maximal $\mathcal{F}$-free subgraph of H. Take $H = K_n$, and $\mathcal{F}$ a family of graphs. Following the definition of the triangle free game of Füredi, Reimer and Seress [3], and building on the notation of West[11], we define the *$\mathcal{F}$-saturation game* as follows.

Two players, *Prolonger* and *Shortener*, build a sequence of graphs G_i. Set $G_0 = \overline{K_n}$, the empty graph on n vertices. The last element of the sequence has index t^* if G_{t^*} is an $\mathcal{F}$-saturated subgraph of H. Otherwise, at time $2t$, Prolonger chooses an edge uv of H not in G_{2t} such that $G_{2t} \cup uv$ is $\mathcal{F}$-free, and we set $G_{2t+1} = G_{2t} \cup uv$. Similarly for Shortener at time $2t + 1$. Prolonger's goal is to maximise t^*, whilst Shortener wishes to minimise t^*. We refer to this game as $\mathcal{G}(H; \mathcal{F})$; for our games, the identity of the starting player has little effect. We look for the value of t^* under optimal play by both players, called the *score* of $\mathcal{G}(H; \mathcal{F})$, denoted $\mathcal{G}(H; \mathcal{F})$ as well. Let $\mathcal{G}(H; F) := \mathcal{G}(H; \{F\})$.

Füredi, Reimer and Seress [3] concentrate on the game $\mathcal{G}(K_n, K_3)$. They exhibit a strategy for Prolonger which demonstrates that $\mathcal{G}(K_n, K_3) \geq (\frac{1}{2} + o(1))n \log_2 n$. They attribute to Erdős a lost proof that Shortener has a strategy showing $\mathcal{G}(K_n, K_3) \leq \frac{n^2}{5}$. To our knowledge, attempts to give a new proof have failed. Motivated by these results, we study the games where $\mathcal{F} = \{P_k\}$, $\mathcal{F}$ is the class of all trees on k vertices or $\mathcal{F} = \{K_{1,k}\}$.

For $\mathcal{F}$ a family of digraphs, we say that a digraph G is *$\mathcal{F}$-homomorphism-free* if for all $F \in \mathcal{F}$, there is no homomorphism $F \to G$. So we can define the *directed $\mathcal{F}$-homomorphism-saturation game* analogously with notation $\mathcal{G}_{\mathrm{dir}}(H; \mathcal{F})$ for the game and also for the score. For $\mathcal{F}$ a family of digraphs, we say a graph G is *$\mathcal{F}$-orientation-free* if there is an orientation G' of G such that G' contains no $F \in \mathcal{F}$ as a subdigraph. So we have the *$\mathcal{F}$-orientation-saturation game* which we *bias* by allowing Prolonger to take a and then Shortener b consecutive turns and refer to this game as $\mathcal{G}_o^{a,b}(H; \mathcal{F})$.

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