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# Graphs with few trivial critical ideals

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**Abstract**

The critical ideals of a graph are determinantal ideals of the generalized Laplacian matrix associated to a graph. Let $\Gamma_{\leq i}$ denote the set of simple connected graphs with at most $i$ trivial critical ideals. The main goal is to obtain a characterization of the graphs in $\Gamma_{\leq 3}$ with clique number equal to 2, and the graphs in $\Gamma_{\leq 3}$ with clique number equal to 3. This shows that there exists a strong connection between the structural properties of the graph (like the clique number and the stability number) with its critical ideals.

*Keywords:* Critical ideal, generalized Laplacian matrix, forbidden induced subgraph.

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# 1 Introduction

Given a graph without loops $G = (V, E)$, the *Laplacian matrix* $L(G)$ of $G$ is the matrix with rows and columns indexed by the vertices of $G$ given by

$$L(G)_{uv} = \begin{cases} \deg_G(u) & \text{if } u = v, \\ -m_{uv} & \text{otherwise,} \end{cases}$$

where $\deg_G(u)$ denote the degree of $u$, and $m_{uv}$ denote the number of edges from $u$ to $v$. By considering the Laplacian matrix of a connected graph $G$ as a linear operator on $\mathbb{Z}^n$, the *critical group* $K(G)$ of $G$ is the torsion part of the cokernel of $L(G)$. The critical group has been studied intensively over the last 30 years on several contexts: the *group of components* [13], the *Picard group* [5,6], the *Jacobian group* [5,6], the *sandpile group* [1], *chip-firing game* [6,14], or *Laplacian unimodular equivalence* [11,15].

It is known (see [12, Theorem 3.9]) that the critical group of a connected graph $G$ with $n$ vertices can be described as follows:

$$K(G) \cong \mathbb{Z}_{f_1} \oplus \mathbb{Z}_{f_2} \oplus \cdots \oplus \mathbb{Z}_{f_{n-1}},$$

where $f_1, f_2, ..., f_{n-1}$ are positive integers with $f_i \mid f_j$ for all $i \leq j$. These integers are called *invariant factors* of the Laplacian matrix of $G$. If $\Delta_i(G)$ is the *greatest common divisor* of the $i$-minors of the Laplacian matrix $L(G)$ of $G$, then the *i-th* invariant factor $f_i$ is equal to $\Delta_i(G)/\Delta_{i-1}(G)$, where $\Delta_0(G) = 1$.

**Definition 1.1** *Given an integer $k$, let $f_k(G)$ be the number of invariant factors of the Laplacian matrix of $G$ equal to $k$.*

The computation of the invariant factors of the Laplacian matrix is an important technique used in the understanding of the critical group. For instance, several researchers addressed the question of how often the critical group is cyclic, that is, if $f_1(G)$ denote the number of invariant factors equal to 1, then the question is how often $f_1(G)$ is equal to $n-2$ or $n-1$? In [13] and [17] D. Lorenzini and D. G. Wagner, based on numerical data, suggest we could expect to find a substantial proportion of graphs having a cyclic critical group. Based on this, D. G. Wagner conjectured [17] that almost

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