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Toward a 6/5 Bound for the Minimum Cost 2-Edge Connected Subgraph Problem [3]

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Abstract

Given a complete graph $K_n = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$, the problem *2EC* is that of finding a 2-edge connected spanning multi-subgraph of K_n of minimum cost. The integrality gap $\alpha\mathit{2EC}$ of the linear programming relaxation $\mathit{2EC}^{\mathrm{LP}}$ for *2EC* has been conjectured to be $\frac{6}{5}$, although currently we only know that $\frac{6}{5} \leq \alpha\mathit{2EC} \leq \frac{3}{2}$. In this paper, we explore the idea of using the structure of solutions for $\mathit{2EC}^{\mathrm{LP}}$ and the concept of convex combination to obtain improved approximation algorithms for *2EC* and bounds for $\alpha\mathit{2EC}$. We focus our efforts on a family J of half-integer solutions that appear to give the largest integrality gap for $\mathit{2EC}^{\mathrm{LP}}$. We successfully show that the conjecture $\alpha\mathit{2EC} = \frac{6}{5}$ is true for any cost functions optimized by some $x^* \in J$. Our methods are constructive and thus also provide a $\frac{6}{5}$-approximation algorithm for *2EC* for these special cases.

Keywords: minimum cost 2-edge connected subgraph problem, approximation algorithm, integrality gap.

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1 Introduction

The *2-edge connected subgraph problem* (*2EC*) is that of finding a minimum cost 2-edge connected spanning multi-subgraph of the complete graph $K_n = (V, E)$ with costs $c \in \mathbb{R}^E_{\geq 0}$. This NP-hard problem has many important applications in network design. Currently, a 2-approximation algorithm exists for general *2EC* [4], and a $\frac{3}{2}$-approximation algorithm exists for metric c [3].

For $e \in E$, letting x_e represent the number of copies of e in the *2EC* solution, *2EC* can be formulated as an integer linear program (LP) as follows:

$$\begin{array}{lll} \text{Minimize} & cx & \\ \text{Subject to} & \sum(x_{ij} : i \in S, j \notin S) \geq 2 & \text{for all } \emptyset \subset S \subset V, \\ & x_e \geq 0, \text{ and integer} & \text{for all } e \in E. \end{array} \tag{1}$$

The LP relaxation of *2EC*, denoted by $2EC^{\mathrm{LP}}$, is obtained by relaxing the integer requirement in (1). We use OPT(*2EC*) (resp. OPT($2EC^{\mathrm{LP}}$)) to denote the optimal value of *2EC* (resp. $2EC^{\mathrm{LP}}$). Also, given any feasible solution x^* for $2EC^{\mathrm{LP}}$, its *support graph* G_{x^*} is defined to be the subgraph of K_n obtained by taking all edges $e \in E$ for which $x^*_e > 0$.

We are interested in the *integrality gap* $\alpha 2EC$ for $2EC^{\mathrm{LP}}$, which is the worst case ratio between OPT(*2EC*) and OPT($2EC^{\mathrm{LP}}$). This gives a measure of the quality of the lower bound provided by $2EC^{\mathrm{LP}}$. Even though *2EC* has been intensively studied, little is known about $\alpha 2EC$, except that $\frac{6}{5} \leq \alpha 2EC \leq \frac{3}{2}$ [1]. Since it has been conjectured that $\alpha 2EC = \frac{6}{5}$ [1], a natural next step is to study $\alpha 2EC$ for some interesting class of cost functions.

We investigate $\alpha 2EC$ for the set of cost functions optimized at a particular family of feasible solutions for $2EC^{\mathrm{LP}}$. A feasible solution x^* for $2EC^{\mathrm{LP}}$ is called a *half-integer solution* if $x^*_e \in \{0, \frac{1}{2}, 1\}$ for all $x^*_e \in E$, and it is called *degree-tight* if $\sum_{uv}(x^*_{uv} : u \in V) = 2$ for all $v \in V$. Finally, a degree-tight half-integer solution is called a *half-triangle solution* if the edges in the support graph G_{x^*} corresponding to $x^*_e = \frac{1}{2}$ (called *half-edges*) form disjoint 3-cycles (called *half-triangles*) joined by paths of edges of value 1 (called *1-paths*).

The half-triangle solutions are of interest for studies of $\alpha 2EC$ as there is evidence that $\frac{\mathrm{OPT}(2EC)}{\mathrm{OPT}(2EC^{\mathrm{LP}})}$ is greatest for cost functions optimized at such solutions (see [1], [2]). For example, the largest such ratio known is asymptotically $\frac{6}{5}$, and comes from an infinite family of half-triangle solutions [1]. Also, in a computational study which found $\alpha 2EC$ exactly for all K_n up to $n = 12$, $\alpha 2EC$ was given by a half-triangle solution for all values of n [1].