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The Bruhat order on conjugation-invariant sets of involutions in the symmetric group

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Abstract

Let I_n be the set of involutions in the symmetric group S_n, and for $A \subseteq \{0, 1, \ldots, n\}$, let

$$F_n^A = \{\sigma \in I_n \mid \sigma \text{ has } a \text{ fixed points for some } a \in A\}.$$

We give a complete characterisation of the sets A for which F_n^A, with the order induced by the Bruhat order on S_n, is a graded poset. In particular, we prove that $F_n^{\{1\}}$ (i.e., the set of involutions with exactly one fixed point) is graded, which settles a conjecture of Hultman in the affirmative. When F_n^A is graded, we give its rank function. We also give a short new proof of the EL-shellability of $F_n^{\{0\}}$ (i.e., the set of fixed point-free involutions), which was recently proved by Can, Cherniavsky, and Twelbeck.

Keywords: Bruhat order, symmetric group, involution, conjugacy class, graded poset, EL-shellability

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1 Introduction

Partially ordered by the Bruhat order, the symmetric group S_n is a graded poset whose rank function is given by the number of inversions, and Edelman [4] proved that it is EL-shellable. Richardson and Springer [10] proved that the set I_n of involutions in S_n and the set F_n^0 of fixed point-free involutions are graded. Incitti [9] proved that the rank function of I_n can be expressed as the average of the number of inversions and the number of exceedances, and that I_n is EL-shellable. Hultman [8] studied (in a more general setting, which we shall describe shortly) F_n^0 and F_n^1, the set of involutions with exactly one fixed point. It follows that F_n^0 is graded and Hultman conjectured that the same is true for F_n^1. Can, Cherniavsky, and Twelbeck [3] recently proved that F_n^0 is EL-shellable.

We consider the following generalisation. For $a \in \{0, 1, \ldots, n\}$, let F_n^a be the conjugacy class in S_n consisting of the involutions with a fixed points, and for $A \subseteq \{0, 1, \ldots, n\}$, let

$$F_n^A = \bigcup_{a \in A} F_n^a.$$

Both I_n and F_n^A are regarded as posets with the order induced by the Bruhat order on S_n. Note that

$$F_n^A = \{\sigma \in I_n \mid \sigma \text{ has } a \text{ fixed points for some } a \in A\}.$$

Also note that for all elements in I_n, the number of fixed points is congruent to n modulo 2. Hence, we may assume that all members of A have the same parity as n.

Depicted in Figures 1 and 2, are the Hasse diagrams of I_4, F_4^0, and F_4^2.

Our main result is a complete characterisation of the sets A for which F_n^A is graded. In particular, we prove that F_n^1 is graded.

Informally, F_n^A is graded precisely when $A - \{n\}$ is empty or an "interval," which may consist of a single element if it is 0, 1, or $n-2$. The following theorem, which is our main result, makes the above precise. It also gives the rank function of F_n^A when it exists.

Theorem 1 *The poset F_n^A is graded if and only if $A - \{n\} = \emptyset$ or $A - \{n\} = \{a_1, a_1 + 2, \ldots, a_2\}$ with $a_1 \in \{0, 1\}$, $a_2 = n - 2$, or $a_2 - a_1 \geq 2$. Furthermore, when F_n^A is graded, its rank function ρ is given by*

$$\rho(\sigma) = \frac{\operatorname{inv}(\sigma) + \operatorname{exc}(\sigma) - n + \tilde{a}}{2} + \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{otherwise,} \end{cases}$$