Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in DISCRETE MATHEMATICS
Electronic Notes in Discrete Mathematics 49 (2015) 221–231
www.elsevier.com/locate/endm

On the zone of a circle in an arrangement of lines

Gabriel Nivasch [1]

Department of Computer Science and Mathematics, Ariel University, Ariel, Israel

Abstract

Let $\mathcal{L}$ be a set of n lines in the plane, and let C be a convex curve in the plane, like a circle or a parabola. The *zone* of C in $\mathcal{L}$, denoted $\mathcal{Z}(C,\mathcal{L})$, is defined as the set of all faces in the arrangement $\mathcal{A}(\mathcal{L})$ that are intersected by C. Edelsbrunner et al. (1992) showed that the complexity (total number of edges or vertices) of $\mathcal{Z}(C,\mathcal{L})$ is at most $O(n\alpha(n))$, where α is the inverse Ackermann function, by translating the sequence of edges of $\mathcal{Z}(C,\mathcal{L})$ into a sequence S that avoids the subsequence *ababa*. Whether the worst-case complexity of $\mathcal{Z}(C,\mathcal{L})$ is only linear is a longstanding open problem.

In this paper we provide evidence that, if C is a circle or a parabola, then the zone of C has at most linear complexity: We show that a certain configuration of segments with endpoints on C is impossible. As a consequence, the Hart–Sharir sequences, which are essentially the only known way to construct *ababa*-free sequences of superlinear length, cannot occur in S.

Hence, if it could be shown that every family of superlinear-length, *ababa*-free sequences must eventually contain all Hart–Sharir sequences, that would settle the zone problem for a circle/parabola.

Keywords: arrangement, zone, Davenport–Schinzel sequence, inverse Ackermann function

[1] Email: `gabrieln@ariel.ac.il`

http://dx.doi.org/10.1016/j.endm.2015.06.032

1 Introduction

Let $\mathcal{L}$ be a set of n lines in the plane. The *arrangement* of $\mathcal{L}$, denoted $\mathcal{A}(\mathcal{L})$, is the partition of the plane into vertices, edges, and faces induced by $\mathcal{L}$. Let C be another object in the plane. The *zone* of C in $\mathcal{L}$, denoted $\mathcal{Z}(C, \mathcal{L})$, is defined as the set of all faces in $\mathcal{A}(\mathcal{L})$ that are intersected by C. The *complexity* of $\mathcal{Z}(C, \mathcal{L})$ is defined as the total number of edges, or vertices, in it.

The celebrated *zone theorem* states that, if C is another line, then $\mathcal{Z}(C, \mathcal{L})$ has complexity $O(n)$ (Chazelle et al. [3]; see also Edelsbrunner et al. [5], Matoušek [12]).

If C is a convex curve, like a circle or a parabola, then $\mathcal{Z}(C, \mathcal{L})$ is known to have complexity $O(n\alpha(n))$, where α is the very-slow-growing inverse Ackermann function (Edelsbrunner et al. [5]; see also Bern et al. [2], Sharir and Agarwal [21]). More specifically, the *outer zone* of $\mathcal{Z}(C, \mathcal{L})$ (the part that lies outside the convex hull of C) is known to have complexity $O(n)$, whereas the complexity of the *inner zone* is only known to be $O(n\alpha(n))$. Whether the complexity of the inner zone is linear as well is a longstanding open problem [2,21].

In this paper we make progress towards proving that the inner zone of a circle, or a parabola, in an arrangement of lines has linear complexity. The problem is more naturally formulated with a circle, but a parabola is easier to work with. Therefore, throughout most of this paper we will take for concreteness C to be the parabola $y = x^2$. In the full version of this paper we show how to modify our argument for the case of a circle.

1.1 Davenport–Schinzel sequences and their generalizations

Let S be a finite sequence of symbols, and let $s \geq 1$ be a parameter. Then S is called a *Davenport–Schinzel sequence of order s* if every two adjacent symbols in S are distinct, and if S does not contain any alternation $a \cdots b \cdots a \cdots b \cdots$ of length $s+2$ for two distinct symbols $a \neq b$. Hence, for $s = 1$ the "forbidden pattern" is aba, for $s = 2$ it is $abab$, for $s = 3$ it is $ababa$, and so on.

The maximum length of a Davenport–Schinzel sequence of order s that contains only n distinct symbols is denoted $\lambda_s(n)$. For $s \leq 2$ we have $\lambda_1(n) = n$ and $\lambda_2(n) = 2n - 1$. However, for fixed $s \geq 3$, $\lambda_s(n)$ is slightly superlinear in n.

The case $s = 3$ is the one most relevant to us. Hart and Sharir [7] constructed a family of sequences that achieve the lower bound [2] $\lambda_3(n) \geq$

[2] See [13] on how to avoid losing a factor of 2 in the interpolation step.