Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in DISCRETE MATHEMATICS
Electronic Notes in Discrete Mathematics 49 (2015) 419–424 www.elsevier.com/locate/endm

Ramsey numbers of ordered graphs

Martin Balko [1,2] Josef Cibulka [1,3] Karel Král [1,4]

Department of Applied Mathematics
Charles University, Faculty of Mathematics and Physics
Malostranské nám. 25, 118 00 Praha 1, Czech Republic

Jan Kynčl [1,5]

Department of Applied Mathematics and Institute for Theoretical Computer Science
Charles University, Faculty of Mathematics and Physics
Malostranské nám. 25, 118 00 Praha 1, Czech Republic
Alfréd Rényi Institute of Mathematics
Reáltanoda u. 13-15, Budapest 1053, Hungary
École Polytechnique Fédérale de Lausanne, Chair of Combinatorial Geometry
EPFL-SB-MATHGEOM-DCG, Station 8, CH-1015 Lausanne, Switzerland

Abstract

An ordered graph is a graph together with a total ordering of its vertices. We study *ordered Ramsey numbers*, the analogue of Ramsey numbers for ordered graphs.

In contrast with the case of unordered graphs, we show that there are ordered matchings whose ordered Ramsey numbers are super-polynomial in the number of vertices.

We also prove that ordered Ramsey numbers are polynomial in the number of vertices of the given ordered graph $\mathcal{G}$ if $\mathcal{G}$ has constant degeneracy and constant interval chromatic number or if $\mathcal{G}$ has constant bandwidth. The latter result answers positively a question of Conlon, Fox, Lee, and Sudakov.

http://dx.doi.org/10.1016/j.endm.2015.06.059

For a few special classes of ordered graphs, we give asymptotically tight bounds for their ordered Ramsey numbers. For so-called monotone cycles we compute their ordered Ramsey numbers exactly.

Keywords: ordered graph, Ramsey number, ordered Ramsey number

---

1 Introduction

Ramsey's theorem states that for every graph G, there exists a number N such that every 2-coloring of the edges of K_N contains a monochromatic copy of G. The least such N is called the *Ramsey number of* G and we denote it by $\mathrm{R}(G)$.

In this paper, we study an analogue of Ramsey's theorem for graphs with ordered vertex sets. In particular, we focus on the effects of different vertex orderings on the Ramsey numbers of given graphs.

An *ordered graph* $\mathcal{G}$ is a pair $(G, \prec)$ where G is a graph and $\prec$ is a total ordering of its vertex set. We say that two ordered graphs $(G, \prec_1)$ and $(H, \prec_2)$ are *isomorphic* if G and H are isomorphic via a one-to-one correspondence $g\colon V(G) \to V(H)$ that also preserves the orderings $\prec_1$ and $\prec_2$. Note that there is only one ordered complete graph $\mathcal{K}_n$ up to isomorphism. An ordered graph $(H, \prec_1)$ is an *ordered subgraph* of $(G, \prec_2)$ if H is a subgraph of G and $\prec_1$ is a suborder of $\prec_2$.

For an ordered graph $\mathcal{G}$, we denote by $\overline{\mathrm{R}}(\mathcal{G})$ the smallest number N such that every 2-coloring of the edges of $\mathcal{K}_N$ contains a monochromatic copy of $\mathcal{G}$ as an ordered subgraph. The number $\overline{\mathrm{R}}(\mathcal{G})$ is called the *ordered Ramsey number of* $\mathcal{G}$. Since $\mathcal{K}_n$ is unique up to isomorphism, for every ordered graph $\mathcal{G} = (G, \prec)$ with n vertices we have $\mathrm{R}(G) \le \overline{\mathrm{R}}(\mathcal{G}) \le \mathrm{R}(K_n)$.

---

[1] The first and the fourth author were supported by the grants GAUK 1262213 of the Grant Agency of Charles University and by the project CE-ITI (GAČR P202/12/G061) of the Czech Science Foundation. The first author was partially supported by the Charles University Grant SVV-2015-260223. The third author acknowledges the support from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n. 616787. The fourth author was also supported by ERC Advanced Research Grant no 267165 (DISCONV) and by Swiss National Science Foundation Grants 200021-137574 and 200020-14453. Part of the research was conducted during the DIMACS REU 2013 program.

[2] Email: `balko@kam.mff.cuni.cz`

[3] Email: `cibulka@kam.mff.cuni.cz`

[4] Email: `kralka@kam.mff.cuni.cz`

[5] Email: `kyncl@kam.mff.cuni.cz`