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On Minimum Bisection and Related Partition Problems in Graphs with Bounded Tree Width

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Abstract

Minimum Bisection denotes the NP-hard problem to partition the vertex set of a graph into two sets of equal sizes while minimizing the number of edges between these two sets. We consider this problem in bounded degree graphs with a given tree decomposition $(T, \mathcal{X})$ and prove an upper bound for their minimum bisection width in terms of the structure and width of $(T, \mathcal{X})$. When $(T, \mathcal{X})$ is provided as input, a bisection satisfying our bound can be computed in time proportional to the encoding length of $(T, \mathcal{X})$. Furthermore, our result can be generalized to k-section, which is known to be APX-hard even when restricted to trees with bounded degree.

Keywords: Minimum Bisection, Minimum k-Section, tree decomposition.

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1 Introduction and Results

Let us first fix some basic terminology. A *cut* $(V_1, V_2, \ldots, V_k)$ in a graph G is a partition of its vertex set. An edge $\{x, y\}$ of G is *cut* by $(V_1, V_2, \ldots, V_k)$ if x and y belong to different sets V_i and V_j. The number of edges cut by $(V_1, V_2, \ldots, V_k)$ is called the *width* of the cut and is denoted by $e(V_1, V_2, \ldots, V_k)$. A k*-section* is a cut $(V_1, V_2, \ldots, V_k)$ such that the sizes of V_i and V_j differ by at most one for all $i, j \in [k]$, where $[k] := \{1, 2, \ldots, k\}$. The *Minimum k-Section Problem* asks to find a *minimum k-section* $(V_1, V_2, \ldots, V_k)$ in a graph G, i.e., a k-section of minimum width among all k-sections in G, and $\mathrm{MinSec}(k, G)$ is defined to be the width of $(V_1, V_2, \ldots, V_k)$. The special case $k = 2$ is also called the *Minimum Bisection Problem*. In what follows, unless stated otherwise, n and $\Delta(G)$ denote the number of vertices and the maximum degree of the considered graph G, respectively.

1.1 Minimum Bisection

Finding a minimum bisection is a famous NP-hard optimization problem [6]. Jansen et al. showed that dynamic programming gives an exact algorithm with running time $\mathcal{O}(2^t n^3)$ when a tree decomposition of width t is provided as input [7]. Thus, the problem becomes polynomially tractable for graphs of bounded tree width. For general graphs, the best known approximation algorithm achieves an approximation ratio of $\mathcal{O}(\log n)$ [9]. Further, the Minimum Bisection Problem restricted to 3-regular graphs is as hard to approximate as its general version [2]. Here, we focus on upper bounds for the minimum bisection width in bounded degree graphs with a given tree decomposition of small width. Lower bounds are more difficult to derive and only few are known. One example is the spectral bound $\mathrm{MinSec}(2, G) \geq \frac{1}{4}\lambda_2 n$, where λ_2 denotes the second eigenvalue of the Laplacian of G [8].

In [4], we have shown that for every tree T

$$\mathrm{MinSec}(2, T) \;\leq\; \frac{8\Delta(T)}{\mathrm{diam}^*(T)}, \tag{1}$$

where $\mathrm{diam}^*(T) := (\mathrm{diam}(T) + 1)/n$ denotes the *relative diameter* of the tree T, i.e., the fraction of vertices of T on a longest path in T. This implies that every tree with linear diameter and bounded maximum degree allows a bisection of constant width. In general, every tree with bounded degree allows a bisection of width $\mathcal{O}(\log_2 n)$ and the perfect ternary tree shows that this is tight up to a constant factor.