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Uniform Linear Embeddings of Spatial Random Graphs

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Abstract

In a random graph with a spatial embedding, the probability of linking to a particular vertex v decreases with distance, but the rate of decrease may depend on the particular vertex v, and on the direction in which the distance increases. In this article, we consider the question when the embedding can be chosen to be uniform, so the probability of a link between two vertices depends only on the distance between them. We give necessary and sufficient conditions for the existence of a uniform linear embedding (embedding into a one-dimensional space) for spatial random graphs where the link probability can attain only a finite number of values.

Keywords: spatial graph model, linear embedding, random graph, geometric graph.

1 Introduction

In the study of large, real-life networks such as on-line social networks and hyperlinked "big data" networks, biological networks and neural connection networks, link formation is often modelled as a stochastic process. The underlying assumption of the link formation process is that vertices have an *a priori*

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identity and relationship to other vertices, which informs the link formation. These identities and relationships can be captured through an embedding of the vertices in a metric space, in such a way that the distance between vertices in the space reflects the similarity or affinity between the identities of the vertices. Link formation is assumed to occur mainly between vertices that have similar identities, and thus are closer together in the metric space. We take as our point of departure a very general stochastic graph model that fits the broad concept of graphs stochastically derived from a spatial layout, along the same principles as described above. We refer to this model as a *spatial random graph*. In a spatial random graph, vertices are embedded in a metric space, and the *link probability* between two vertices depends on this embedding in such a way that vertices that are close together in the metric space are more likely to be linked.

The concept of a spatial random graph allows for the possibility that the link probability depends on the spatial position of the vertices, as well as their metric distance. Thus, in the graph we may have tightly linked clusters for two different reasons. On the one hand, such clusters may arise when vertices are situated in a region where the link probability is generally higher. On the other hand, clusters can still arise when the link probability function is *uniform*, in the sense that the probability of a link between two vertices depends only on their distance, and not on their location. In this case, tightly linked clusters can arise if the distribution of vertices in the metric space is inhomogeneous. A cluster in the graph then points to a corresponding tightly packed cluster in the metric space. The central question addressed in this paper is how to recognize spatial random graphs with a uniform link probability function.

Here we study a one-dimensional spatial model where the metric space is $[0,1]$. Let $\mathcal{W}_0$ be the class of symmetric measurable functions from $[0,1]^2$ to $[0,1]$, and let $w \in \mathcal{W}_0$. The *w-random graph*, $G(n,w)$ is the graph with vertex set V of n points chosen uniformly from the metric space $[0,1]$. Then two vertices $x, y \in V$ are linked with probability $w(x,y)$. The w-random graphs was first introduced in [3]. For the graph $G(n,w)$ to correspond to a notion of spatial random graph, w must be such that points closer together have a higher probability of being linked. This implies that for $x < y$ the value of $w(x,y)$ decreases whenever x decreases or y increases. We call $w \in \mathcal{W}_0$ a *diagonally increasing function* if it satisfies the above property.

The notion of diagonally increasing functions and our interpretation of spatial random graphs were first given at a previous work. See [1]. Such random models have several real life applications (see [2], and [4] for applications in social networks, and [5] for neuroscience). In [1], a graph parameter