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Electronic Notes in DISCRETE MATHEMATICS
Electronic Notes in Discrete Mathematics 49 (2015) 569–575 www.elsevier.com/locate/endm

Evolution of high-order connected components in random hypergraphs

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Abstract

We consider high-order connectivity in k-uniform hypergraphs defined as follows: Two j-sets are j-connected if there is a walk of edges between them such that two consecutive edges intersect in at least j vertices. We describe the evolution of j-connected components in the k-uniform binomial random hypergraph $\mathcal{H}^k(n,p)$. In particular, we determine the asymptotic size of the giant component shortly after its emergence and establish the threshold at which $\mathcal{H}^k(n,p)$ becomes j-connected with high probability. We also obtain a hitting time result for the related random hypergraph process $\{\mathcal{H}^k(n,M)\}_M$ – the hypergraph becomes j-connected exactly at the moment when the last isolated j-set disappears. This generalises well-known results for graphs and vertex-connectivity in hypergraphs.

Keywords: Random hypergraphs, high-order connectivity, hitting time, giant component, phase transition.

[1] The authors are supported by Austrian Science Fund (FWF): P26826, W1230.
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http://dx.doi.org/10.1016/j.endm.2015.06.077
1571-0653/

1 Evolution of random graphs

The theory of random graphs was founded in the late 1950s by Erdős and Rényi describing the evolution of the random graph process $\{G(n, M)\}_M$. The vertex set of this process is $[n] := \{1, \ldots, n\}$ and initially there are no edges present. In each step of the process, add an edge between a pair of vertices chosen uniformly at random amongst all pairs of vertices that do not already form an edge. In the early stages of this process, all connected components are small and then, within very short time, they merge into a single component of linear size – the *giant component*. This remarkable phenomenon, first proved in [8], is known as the *phase transition* of the random graph process $\{G(n, M)\}_M$.

It is often more convenient to analyse the binomial random graph $G(n, p)$: The vertex set is $[n]$ and every pair of vertices is connected by an edge with probability p independently. Incorporating various strengthenings the phase transition can be summarised as follows. (All asymptotic statements are with respect to $n \to \infty$ and by *whp* we abbreviate "with probability $\to 1$".)

Theorem 1.1 (Bollobás [2]; Łuczak [11]) *Let $\varepsilon = \varepsilon(n) > 0$ be a real function satisfying $\varepsilon \to 0$ and $\varepsilon^3 n \to \infty$.*

(i) *If $p = \frac{1-\varepsilon}{n}$, then whp all components in $G(n, p)$ have size $O(\varepsilon^{-2} \log(\varepsilon^3 n))$;*

(ii) *If $p = \frac{1+\varepsilon}{n}$, then whp the size of the largest component in $G(n, p)$ is $(1 \pm o(1))2\varepsilon n$, while all other components have size $O(\varepsilon^{-2} \log(\varepsilon^3 n))$.*

As we continue to add edges one by one, more and more components are consumed by the giant component and eventually the graph becomes connected. In fact, Bollobás and Thomason [4] showed that this happens precisely at the moment when the last isolated vertex disappears – thereby relating a *global graph property* to its minimal *local* obstruction. Denote the *hitting time* of connectivity by τ_{c}, i.e. τ_{c} is the minimal M such that $G(n, M)$ is connected, and the hitting time for the disappearance of the last isolated vertex by τ_{i}.

Theorem 1.2 (Bollobás & Thomason [4]) *Whp $\tau_{\mathrm{c}} = \tau_{\mathrm{i}}$ for $\{G(n, M)\}_M$.*

2 Evolution of random hypergraphs – Main results

Given an integer $k \geq 2$ a *k-uniform hypergraph* H consists of a set V of vertices and a set E of edges, where each edge contains precisely k vertices. In particular, 2-uniform hypergraphs are simply graphs. Given an integer $1 \leq j \leq k - 1$ we say that two j-sets (sets of j distinct vertices) J and J' are *j-connected* if there is a sequence of edges $e_1, \ldots, e_m$ such that $J \subset e_1$,