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On Spanning Structures in Random Hypergraphs

O. Parczyk [1,2] Y. Person [1,3]

Institut für Mathematik
Goethe-Universität
Robert-Mayer-Str. 10, 60325 Frankfurt am Main, Germany

Abstract

In this note we adapt a general result of Riordan [*Spanning subgraphs of random graphs*, Combinatorics, Probability & Computing **9** (2000), no. 2, 125–148] from random graphs to random r-uniform hypergaphs. We also discuss several spanning structures such as cube-hypergraphs, lattices, spheres and Hamilton cycles in hypergraphs.

Keywords: Random hypergraphs, spanning structures, threshold functions.

1 Introduction

Finding spanning subgraphs is a well studied problem in random graph theory, see e.g. the following monographs on random graphs [4,14]. In the case of hypergraphs not so much is known and it is natural to study the corresponding problems for hypergraphs.

[1] This research was supported by DFG grant PE 2299/1-1.
[2] Email: `parczyk@math.uni-frankfurt.de`
[3] Email: `person@math.uni-frankfurt.de`

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An r-uniform hypergraph H is a tuple (V, E), where V is its vertex set and $E \subseteq \binom{V}{r}$ the set of edges in H. Further we write $\deg(v)$ for the degree of a vertex v in H: $\deg(v) := |\{e : v \in e\}|$, and $\Delta(H)$ denotes the maximum vertex degree in H, i.e. $\Delta(H) := \max_{v \in V} \deg(v)$. We will consider two models of r-uniform random hypergraphs $\mathcal{H}^{(r)}(n, p)$ and $\mathcal{H}^{(r)}(n, m)$. Formally, $\mathcal{H}^{(r)}(n, p)$ is the probability space of all labelled r-uniform hypergraphs with the vertex set $[n]$ where each edge $e \in \binom{[n]}{r}$ is chosen independently of all the other edges with probability p. Similarly one defines $\mathcal{H}^{(r)}(n, m)$ as the probability space of all labelled r-uniform hypergraphs with the vertex set $[n]$ and exactly m edges and considers a uniform measure.

We will shortly write $\mathcal{H}$ for a random graph in one of these classes and all probabilities are with respect to the corresponding model. For $r = 2$ these are the standard models $G(n, p)$ and $G(n, m)$ respectively.

Let $H = H^{(i)}$ be a sequence of fixed r-uniform hypergraphs with n vertices, where $n = n(i) \to \infty$. Then we say that $\mathcal{H}$ contains the graph H asymptotically almost surely (a.a.s.) if the probability that $H^{(i)} \subseteq \mathcal{H}$ tends to 1 as n tends to infinity (here $\mathcal{H} = \mathcal{H}^{(r)}(n, p)$ or $\mathcal{H} = \mathcal{H}^{(r)}(n, m)$). We say that $\hat{p}$ is a threshold function if $\mathbb{P}[H \subseteq \mathcal{H}^{(r)}(n, p)] \to 0$ for $p \ll \hat{p}$ and $\mathbb{P}[H \subseteq \mathcal{H}^{(r)}(n, p)] \to 1$ for $p \gg \hat{p}$ as n tends to infinity. Similarly one defines a threshold function $\hat{m} = \hat{m}(n)$ in the model $\mathcal{H}^{(r)}(n, m)$. It was shown by Bollobás and Thomason [5] that all nontrivial monotone properties have a threshold function. Since subgraph containment is a monotone property it is natural to study the threshold functions for appearance of various structures in random graphs and hypergraphs.

The case of *fixed* (hyper-)graphs was solved by Erdős and Rényi [10](balanced case) and by Bollobás [3]. First spanning structures considered in graphs were perfect matchings [11] and Hamilton cycles [4,17,23]. More recently, the thresholds for the appearance of (bounded degree) spanning trees were studied as well, for the currently best bounds see Montgomery [19,20].

Alon and Füredi [2] studied the question when the random graph $G(n, p)$ contains a given graph G of bounded maximum degree Δ hereby proving the bound $p \geq C(\ln n/n)^{1/\Delta}$ for some absolute constant $C > 0$. In [24] Riordan proved quite a general theorem applicable to various graphs including hypercubes and lattices. Finding thresholds for factors of graphs and hypergraphs was long an open problem where breakthrough was achieved by Johansson, Kahn and Vu [15]. Kahn and Kalai [16] have a general conjecture about the thresholds for the appearance of a given structure (which roughly states that the threshold p with $\mathbb{P}(G \subseteq G(n, p)) = 1/2$ for containment of G is within a factor of $O(\ln n)$ from p_E at which the expected number of copies of G in