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ScienceDirect
Electronic Notes in DISCRETE MATHEMATICS
Electronic Notes in Discrete Mathematics 49 (2015) 699–703
www.elsevier.com/locate/endm

Equitable colorings of non-uniform simple hypergraphs

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Abstract

The paper is devoted to the combinatorial problem concerning equitable colorings of non-uniform simple hypergraphs. Let $H = (V, E)$ be a hypergraph, a coloring with r colors of its vertex set V is called equitable if it is proper (i.e. none of the edges is monochromatic) and the cardinalities of the color classes differ by at most one. We show that if H is a simple hypergraph with minimum edge-cardinality n and

$$\sum_{e \in E} r^{1-|e|} \leqslant c\sqrt{n},$$

for some absolute constant $c > 0$, then H has an equitable r-coloring.

Keywords: Hypergraphs, colourings, Property B.

[1] This work was supported by Russian Foundation of Fundamental Research (grant 15-01-03530-) and by the grant of the President of Russian Federation MK-692.2014.1
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http://dx.doi.org/10.1016/j.endm.2015.06.094

1 Introduction

The paper deals with some aspects of the well-known problem of Erdős and Lovász concerning colorings of non-uniform hypergraphs.

Let us begin our review with well-known extremal problems concerning colorings of hypergraphs is the classical Property B problem stated by P. Erdős and A. Hajnal. The problem is to find the value $m(k)$ equal to the minimum possible number of edges in a k-uniform hypergraph which is not 2-colorable (see the survey [1] for the details). We shall recall a fragment from its history.

In 1973 Erdős and L. Lovász in their seminal paper [2] conjectured that

$$\frac{m(k)}{2^k} \to \infty \quad ask \to \infty.$$

Furthermore, they formulated a stronger conjecture concerning non-uniform hypergraphs. For a hypergraph $H = (V, E)$, let $f(H)$ denote the function

$$f(H) = \sum_{e \in E} 2^{-|e|}.$$

Erdős and Lovász proposed to consider the value $f(k)$ equal to the minimum possible value of $f(H)$ where H is 3-chromatic hypergraph with minimum edge-cardinality k. They conjectured that

$$f(k) \to \infty \quad ask \to \infty.$$

Both conjectures were proved by J. Beck in 1977–78. He proved that $m(k) \geqslant 2^k k^{1/3-o(1)}$, but his lower bound for $f(k)$ was much weaker. Using the function $\log^*(x)$ Beck's result (see [3]) can be formulated as follows:

(1) $$f(k) \geqslant \frac{\log^*(k) - 100}{7},$$

where $\log^*(k)$ is the inverse function for the tower of twos of height k. Thus this inequality proves the conjecture of Erdős and Lovász, but grows very slowly. In [4] D. Shabanov improved Beck's condition (which guarantees r-colorability in terms of $f(H)$) for simple triangle-free hypergraphs. He showed that if $H = (V, E)$ is triangle-free simple hypergraph with minimum edge-cardinality k and

(2) $$f_r(H) = \sum_{e \in E} r^{1-|e|} \leqslant \frac{1}{2}\left(\frac{k}{\ln k}\right)^{2/3},$$