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# Sudoku Rectangle Completion

(Extended Abstract)

Mohammad Mahdian [1]

*Google Research*
*Mountain View, CA, USA*

Ebadollah S. Mahmoodian [2,3]

*Department of Mathematical Sciences*
*Sharif University of Technology*
*Tehran, I.R. IRAN*

**Abstract**

Over the last decade, Sudoku, a combinatorial number-placement puzzle, has become a favorite pastimes of many all around the world. In this puzzle, the task is to complete a partially filled $9 \times 9$ square with numbers 1 through 9, subject to the constraint that each number must appear once in each row, each column, and each of the nine $3 \times 3$ blocks. Sudoku squares can be considered a subclass of the well-studied class of Latin squares. In this paper, we study natural extensions of a classical result on Latin square completion to Sudoku squares. Furthermore, we use the procedure developed in the proof to obtain asymptotic bounds on the number of Sudoku squares of order $n$.

*Keywords:* Sudoku squares, Latin squares, Number of Sudoku squares, Critical sets in Latin squares.

[1] Email: `mahdian@alum.mit.edu`
[2] Email: `emahmood@sharif.edu`
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# 1 Introduction and preliminaries

A Latin square is an $n \times n$ matrix with entries in $1, \ldots, n$ such that each of the numbers 1 to $n$ appears exactly once in each row and in each column. Latin squares are heavily studied combinatorial objects that date back to the time of Euler and probably earlier. A variant of the notion of Latin squares has recently surfaced in the form of a number-placement puzzle called Sudoku. In this puzzle, the task is to complete a partially filled $9 \times 9$ square with numbers 1 through 9 such that in addition to the Latin square conditions, each number appears exactly once in each of the nine $3 \times 3$ blocks. This puzzle was popularized in 1986 by the Japanese puzzle company Nikoli and became an international hit in the 2000s, although similar puzzles have appeared in various publications around the world since late 19th century. The emergence of this puzzle has generated a surge of interest in the mathematical properties of Sudoku squares [6].

In this paper, we study a Sudoku rectangle completion problem similar to a classical result of M. Hall on Latin rectangle completion. We start with the formal definition of the main notions used in this paper. Definitions and notations not given here may be found in standard combinatorics and graph theory textbooks such as [2] and [12].

A Latin square of order $n$ is an $n \times n$ matrix with entries from $[n] = \{1, \ldots, n\}$ that satisfies the following two conditions:

- **row condition**: every element in $[n]$ appears at most once in each row.
- **column condition**: every element in $[n]$ appears at most once in each column.

When $n = k^2$ for an integer $k$, for every $i, j \in [k]$, the $(i, j)$th block of an $n \times n$ matrix $M$ is defined as the set of entries with coordinates in $((i-1)k + x, (j-1)k + y)$ for $x, y \in [k]$. We say that $(i, j)$ are the coordinates of this block. These blocks partition the set of entries in $M$ into $k^2$ submatrices, each of size $k \times k$ and therefore containing $n$ entries. The $i$th row block of $M$ is the union of the blocks at coordinates $(i, j)$ for $j \in [k]$. Similarly, the $j$th column block of $M$ is the union of the blocks at coordinates $(i, j)$ for $i \in [k]$. A Sudoku square of order $n = k^2$ is an $n \times n$ matrix that in addition to the row and column conditions above, satisfies the following condition:

- **block condition**: every element in $[n]$ appears at most once in each block.

A partial Latin (Sudoku) square of order $n$ is an $n \times n$ matrix with entries from $[n] \cup \{*\}$ (with $*$ representing an *empty* entry) that satisfies the row and