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Equivariant Euler characteristics of partition posets

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ABSTRACT

The first part of this paper deals with the combinatorics of equivariant partitions of finite sets with group actions. In the second part, we compute all equivariant Euler characteristics of the Σ_n-poset of non-extreme partitions of an n-set.

1. Introduction

Let G be a finite group, Π a finite G-poset, and $r \geq 1$ a natural number. For any homomorphism $X : \mathbf{Z}^r \to G$, write Π^X for the sub-poset consisting of all elements of Π fixed by all elements in the image of X. The *rth equivariant reduced Euler characteristic* of the G-poset Π was defined by Atiyah and Segal [2] as the normalized sum

$$\widetilde{\chi}_r(\Pi, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbf{Z}^r, G)} \widetilde{\chi}(\Pi^X) \tag{1.1}$$

of the reduced Euler characteristics $\widetilde{\chi}(\Pi^X)$ (Definition 3.1(1)) of the X-fixed sub-poset Π^X as X runs through the set of all homomorphisms of $\mathbf{Z}^r$ to G, or, equivalently, the set of r-tuples of commuting elements of G. Two extreme examples are the following. When the poset $\Pi = \emptyset$ is empty, $\widetilde{\chi}_r(\emptyset, G) = -|\mathrm{Hom}(\mathbf{Z}^r, G)|/|G|$, and when Π has a least or greatest element, $\widetilde{\chi}_r(\Pi, G) = 0$ for all $r \geq 1$.

We are here particularly interested in G-posets of partitions of G-sets. For a finite G-set S, let $\Pi(S)$ denote the G-lattice of partitions of S and $\Pi^*(S) = \Pi(S) - \{\widehat{0}, \widehat{1}\}$ its proper part, the sub-G-poset of

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non-extreme partitions obtained by removing the discrete partition $\widehat{0}$ and the indiscrete partition $\widehat{1}$. The rth equivariant reduced Euler characteristic of the partition G-poset $\Pi^*(S)$ is the normalized sum

$$\widetilde{\chi}_r(\Pi^*(S), G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbf{Z}^r, G)} \widetilde{\chi}(\Pi^*(S)^X) \tag{1.2}$$

of the reduced Euler characteristics of the sub-posets $\Pi^*(S)^X$ of non-extreme X-partitions of S as X ranges over the set of commuting r-tuples of elements of G.

Eq. (1.2) above highlights the relevance of Euler characteristics of G-partitions of G-sets. The first part of this paper, dealing with the combinatorics of posets of G-partitions of G-sets, addresses this issue. The main result here, Theorem 3.9, identifies the reduced Euler characteristic $\widetilde{\chi}(\Pi^*(S)^G)$ as a G-Stirling number of the first kind.

In the second part we compute the equivariant reduced Euler characteristics $\widetilde{\chi}_r(\Pi^*(S), G)$ in the archetypical case where $G = \Sigma_n$ is the symmetric group of degree n and $S = \Sigma_{n-1}\backslash\Sigma_n$ the standard n-element right Σ_n-set. The Σ_n-poset $\Pi^*(\Sigma_{n-1}\backslash\Sigma_n)$ consists of all non-extreme partitions of the n-set. The main result, Theorem 1.3 below, describes the equivariant reduced Euler characteristics $\widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ for all $r \geq 1$ and all $n \geq 1$. (It is convenient to declare $\widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ to mean 1 for all $r \geq 1$ when $n = 1$ even though the equivariant reduced Euler characteristics actually equal -1 in these cases.)

Let π_k, $k \geq 0$, be the multiplicative functions given by $\pi_k(n) = n^k$ for all $n \geq 1$ and ι_2 the multiplicative function given by $\iota_2(n) = n$ if $n = 1, 2, 4, \dots$ is a power of 2 and $\iota_2(n) = 0$ otherwise.

Theorem 1.3. *The rth reduced equivariant Euler characteristic of the Σ_n-poset $\Pi^*(\Sigma_{n-1}\backslash\Sigma_n)$ is*

$$\widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n) = c_r(n)/n, \quad n \geq 1, r \geq 1$$

where the multiplicative function c_r is the Dirichlet inverse

$$c_r = (\iota_2 * \pi_1 * \cdots * \pi_{r-1})^{-1}$$

of the iterated Dirichlet convolution of the function ι_2 and the $r-1$ functions π_k for $0 < k < r$.

The corollary below presents two alternative and more explicit views on the rth equivariant reduced Euler characteristics of Theorem 1.3. Let $b_0 = \varepsilon$ be the multiplicative Dirichlet unit function $\varepsilon = 1, 0, 0, \dots$ and for $r \geq 1$, let $b_r(n)$ and $\lambda_r(n)$ be the multiplicative functions whose values on prime powers $n = p^e$ are

$$b_r(p^e) = (-1)^e p^{\binom{e}{2}} \binom{r}{e}_p, \qquad \lambda_r(p^e) = \binom{e+r-1}{e}_p$$

where $\binom{n}{k}_p$ refers to a p-binomial coefficient (Eq. (6.2)). We note in Corollary 6.8 that $\lambda_r(n)$ is the number of subgroups of $\mathbf{Z}^r$ of index n and that b_r and that λ_r are reciprocal under Dirichlet convolution, $b_r * \lambda_r = \varepsilon$.

Corollary 1.4. *Fix $r \geq 1$.*

(1) $(c_r * \lambda_r)(n) = (-1)^{n+1}$ *for all $n \geq 1$.*
(2) $\sum_{d|n} d\widetilde{\chi}_r(\Pi^*(\Sigma_{d-1}\backslash\Sigma_d), \Sigma_d)\lambda_r(n/d) = (-1)^{n+1}$ *for all $n \geq 1$.*
(3) *The multiplicative function $c_r(n)/n = \widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ takes value $b_{r-1}(2^e) - b_{r-1}(2^{e-1})$ on an even prime power $n = 2^e$, $e > 0$, and value $b_{r-1}(p^e)$ on an odd prime power $n = p^e$, $e \geq 0$.*

The equation of Corollary 1.4(2) provides a recurrence relation for the rth equivariant reduced Euler characteristic function $\widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ when regarding the rth subgroup enumeration function $\lambda_r(n)$ known. Eq. (7.5) makes explicit the fact that the values given in Corollary 1.4(3) completely determine the equivariant reduced Euler characteristics $\widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ for all r and n. See Fig. 2 for concrete numerical values of $c_r(n)/n = \widetilde{\chi}_r(\Pi^*(\Sigma_{n-1}\backslash\Sigma_n), \Sigma_n)$ for small r and n. The proofs of Theorem 1.3 and Corollary 1.4 are in Sections 5 and 7.