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Separating n-point sets from quasi-finite ones via polyhedral surfaces

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ABSTRACT

Let X be an infinite set in $\mathbb{R}^d$ that has no accumulation point. We prove that the following statement holds for each d-dimensional polyhedron Π, i.e., for each bounded part of $\mathbb{R}^d$ generated by a closed polyhedral surface: for any positive integer n, there is a polyhedron similar to Π that contains exactly n points taken from X.

1. Introduction

H. Steinhaus proved that for every natural number n, there is a circle in the plane which contains in its interior exactly n lattice points; see H. Steinhaus [8, problem 24] and R. Honsberger [1, p. 118]. (A lattice point is a point all whose coordinates are integers.) This result is extended in [2] as follows: for any bounded convex domain W in the plane and for any $n > 0$, there is a set similar to W that contains exactly n lattice points. In [3] it is proved that for every (convex or concave) polygon P in the plane, there is a polygon similar to P that contains a given number of lattice points.

A. Schinzel [7] proved that for every natural number $n > 0$, there is a circle in the plane that passes through exactly n lattice points; see also [5]. Similar problems are considered for quadratic curves in [2]. There the authors proved that, e.g., for each $0 \leq n < 5$ there is a parabola passing through exactly n lattice points, but if a parabola passes through five lattice points, then it passes through infinitely many lattice points.

P. Zwolenski [9] considered such Steinhaus-type problems in a more general setting. In a metric space M, a countable subset $X \subset M$ is called *quasi-finite* if every ball in M contains only finitely many

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points of X. P. Zwolenski proved the following for quasi-finite sets in Hilbert spaces: if X is such a set, then there is a dense set Y in this Hilbert space such that for every $y \in Y$ and every integer $n > 0$ there is a ball with center y that contains exactly n points of X.

We consider here the problem of separating a subset of given cardinality from a quasi-finite set in $\mathbb{R}^d$ by a polyhedron. A quasi-finite set in $\mathbb{R}^d$ is also characterized as an infinite subset of $\mathbb{R}^d$ that has no accumulation point. By a *closed polyhedral surface* in $\mathbb{R}^d$, we mean a $(d-1)$-dimensional closed manifold in $\mathbb{R}^d$ that is contained in a union of a finitely many hyperplanes in $\mathbb{R}^d$. By the Jordan–Brouwer separation theorem (see, e.g., [6]), a closed polyhedral surface in $\mathbb{R}^d$ divides $\mathbb{R}^d$ into two components, one bounded and the other one unbounded. The closure of such a bounded component is called a d-dimensional Jordan–Brouwer polyhedron or, shortly, a *polyhedron*. For a polyhedron Π, the closed polyhedral surface that bounds Π is denoted by $\partial\Pi$.

Theorem 1. *Let X be a quasi-finite set in $\mathbb{R}^d$, and Π be a polyhedron in $\mathbb{R}^d$. For any integer $n > 0$, there is a polyhedron similar to Π that contains exactly n points of X.*

The proof is accomplished by showing that there is a continuous deformation $\mathcal{H}(s)$, $0 \le s \le 1$, of a polyhedron such that

(a) for every $s \in [0, 1]$, $\mathcal{H}(s)$ is a polyhedron that is similar to Π,
(b) $|\mathcal{H}(0) \cap X| \le n-1$ and $|\mathcal{H}(1) \cap X| \ge n+1$, and
(c) for every $s \in [0, 1]$, $|\partial\mathcal{H}(s) \cap X| \le 1$.

Condition (c) implies that, when s increases from 0 to 1, $|\mathcal{H}(s) \cap X|$ changes one by one, and hence, by condition (b), there must be an $s_0 \in [0, 1]$ such that $|\mathcal{H}(s_0) \cap X| = n$.

The set of all lattice points in $\mathbb{R}^d$ is clearly a quasi-finite set. Hence we have the following result from [4] as a corollary of Theorem 1.

Corollary 1. *For any polyhedron Π in $\mathbb{R}^d$ and any integer $n > 0$, there is a polyhedron that is similar to Π and contains exactly n lattice points.*

For a polyhedron Π and a real number $\lambda > 0$, the set $\lambda\Pi$ denotes the *homothet of Π with ratio λ*.

Remark 1. If we replace "a polyhedron similar to Π" by "a homothet $\lambda\Pi$" in Theorem 1, then it is no longer true. To see this, consider the case that X is the set of all lattice points in $\mathbb{R}^2$, and Π is a square whose edges are parallel to the coordinate-axes. Then, every homothet $\lambda\Pi$ of Π is also a square with edges parallel to the coordinate-axes, and hence the number of lattice points in $\lambda\Pi$ is either $m \times m$ or $(m-1) \times m$ or $(m-1) \times (m-1)$ for some integer $m > 0$. Since none of these numbers is equal to, e.g., an odd prime p, there is no $\lambda\Pi$ that contains exactly p lattice points.

Theorem 2. *Let Y be a finite point set in $\mathbb{R}^d$ and Π be a polyhedron. If N points of Y lie in the interior of Π, then, for every $0 < n < N$, there is a polyhedron congruent to $\lambda\Pi$ for some $0 < \lambda < 1$ that contains exactly n points of Y.*

Remark 2. If we drop the condition $0 < \lambda < 1$ in this theorem, then the assertion would follow easily from the fact that there is a hyperplane that separates n points of Y.

2. Notation and two lemmas

Throughout this paper, X denotes a quasi-finite set in $\mathbb{R}^d$. As already introduced, Π denotes a (d-dimensional) *polyhedron* in $\mathbb{R}^d$, and $\Sigma = \partial\Pi$ is the closed polyhedral surface that bounds Π. We may suppose that the origin O is an interior point of Π, and that $O \notin X$. We use the following notations: For $A \subset \mathbb{R}^d$, $v \in \mathbb{R}^d$, and $t > 0$,

$$v + A = \{v + x : x \in A\} \quad \text{(the } \textit{translate of } A \textit{ by } v\text{)},$$
$$tA = \{tx : x \in A\} \quad \text{(the } \textit{homothetic copy of } A \textit{ with ratio } t\text{)},$$
$$A^* = \{-x : x \in A\} \quad \text{(the } \textit{set symmetric to } A \textit{ with respect to } O\text{)}.$$