European Journal of Combinatorics 60 (2017) 100–113

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Signed enumeration of upper-right corners in path shuffles

William Kuszmaul [1]

University of Minnesota Duluth, 1049 University Dr., Duluth, MN 55812, United States

ARTICLE INFO

Article history:
Received 12 August 2015
Accepted 20 September 2016
Available online 11 October 2016

ABSTRACT

We resolve a conjecture of Albert and Bousquet-Mélou enumerating quarter-planar walks with fixed horizontal and vertical projections according to their upper-right-corner count modulo 2. In doing this, we introduce a signed upper-right-corner count statistic. We find its distribution over planar walks with any choice of fixed horizontal and vertical projections. Additionally, we prove that the polynomial counting loops with a fixed horizontal and vertical projection according to the absolute value of their signed upper-right-corner count is $(x+1)$-positive. Finally, we conjecture an equivalence between $(x+1)$-positivity of the generating function for upper-right-corner count and signed upper-right-corner count, leading to a reformulation of a conjecture of Albert and Bousquet-Mélou on which their asymptotic analysis of permutations is sortable by two stacks in parallel relies.

1. Introduction

When studying walks on $\mathbb{Z}^2$ with unit steps, it is natural to restrict oneself to walks which project vertically on a fixed vertical path $\mathcal{V}$ and horizontally on a fixed horizontal path $\mathcal{H}$, where $\mathcal{V}$ (resp. $\mathcal{H}$) comprises North and South steps (resp. East and West steps) [1]. Such paths correspond with *shuffles*, or interleavings, of $\mathcal{V}$ and $\mathcal{H}$. If $|\mathcal{V}| = v$ and $|\mathcal{H}| = h$, there are $\binom{v+h}{h}$ such shuffles. It will be convenient to denote East, West, North, and South steps by $\rightarrow$, $\leftarrow$, $\uparrow$, and $\downarrow$.

E-mail address: william.kuszmaul@gmail.com.

1 Present address: Stanford University, P.O. Box 16632, 531 Lasuen Mall, Stanford, CA 94305, USA.

http://dx.doi.org/10.1016/j.ejc.2016.09.007
0195-6698/

Given a path, several natural statistics arise. In this paper, we focus on variants of *peak-count*, which counts the number of NW (i.e., ↰) and ES (i.e., ↴) corners. If the path lies within the first quadrant, then the peak-count is visually the number of peaks pointing away from the origin. Equivalently, peak-count counts occurrences of $\rightarrow\downarrow$ and $\uparrow\leftarrow$ in a shuffle of $\mathcal{V}$ and $\mathcal{H}$. Albert and Bousquet-Mélou introduced peak-count and showed it to have applications in the study of permutations sortable by two stacks in parallel [1]. They also proved that peak-count has the same distribution over shuffles as do several other statistics (Proposition 14 of [1], which was originally observed by Julien Courtiel and Olivier Bernardi independently).

Albert and Bousquet-Mélou studied several generating functions involving peak-count of planar walks in depth. Although very little is known in the case where the path's horizontal and vertical projections are fixed, Albert and Bousquet-Mélou posed the following conjecture, which they attribute to Julien Courtiel [1].

Conjecture 1.1 (*Albert and Bousquet-Mélou, Conjecture (P1) on p. 32 [1]*)**.** *Let $\mathcal{H}$ (resp. $\mathcal{V}$) be a path beginning and ending at the origin of half-length i (resp. j) on the alphabet $\{\rightarrow, \leftarrow\}$ (resp. $\{\uparrow, \downarrow\}$). Then the polynomial that counts walks of the shuffle class of $\mathcal{H}\mathcal{V}$ according to the number of ↰ and ↴ corners takes the value $\binom{i+j}{i}$ when evaluated at -1. Equivalently, the shuffle class of $\mathcal{H}\mathcal{V}$ contains $\binom{i+j}{i}$ more shuffles with even peak-count than with odd peak-count.*

Albert and Bousquet-Mélou note that, among other things, proving the conjecture would eliminate the need for the somewhat lengthy proof of their Proposition 15 [1].

We prove Conjecture 1.1 and extend it to the case where $\mathcal{H}$ and $\mathcal{V}$ are arbitrary words on the alphabets $\{\rightarrow, \leftarrow\}$ and $\{\uparrow, \downarrow\}$ respectively. This corresponds with considering arbitrary walks on the plane, rather than only planar loops.

In order to study peak-count modulo 2, it is sufficient to study what we call *signed peak-count*, which is the number of ↰ corners minus the number of ↴ corners. While signed peak-count and peak-count are guaranteed to share the same parity, it turns out that signed peak-count exhibits nice behavior allowing for it to be completely enumerated.

Theorem 1.2. *Let $\mathcal{V}$ be a word comprising u $\uparrow$'s and d $\downarrow$'s. Let $\mathcal{H}$ be a word comprising r $\rightarrow$'s and l $\leftarrow$'s. Then the number of shuffles of $\mathcal{V}$ and $\mathcal{H}$ with signed peak-count k is*

$$\binom{r+u}{u-k}\binom{l+d}{d+k}.$$

Note that Theorem 1.2 immediately leads to a formula for the difference between the number of even and odd peak-count shuffles of $\mathcal{V}$ and $\mathcal{H}$.

$$\sum_k (-1)^k \binom{r+u}{u-k}\binom{l+d}{d+k}. \tag{1}$$

In certain key cases, Formula (1) simplifies. When $r = l$ and $u = d$ (as is the case in Conjecture 1.1), Formula (1) becomes

$$\sum_k (-1)^k \binom{r+u}{u-k}\binom{r+u}{u+k} = \binom{r+u}{u},$$

by equation (30) of [6].

Additionally, when $r = u$ and $l = d$, Formula (1) becomes

$$\sum_k (-1)^k \binom{2r}{r-k}\binom{2l}{l+k},$$

which is known [6, Equation 29] to be the super Catalan number,

$$S(r,k) = \frac{(2r)!(2k)!}{r!k!(r+k)!}.$$