European Journal of Combinatorics 60 (2017) 114–123

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Conditional expanding bounds for two-variable functions over finite valuation rings

Le Quang Ham [a], Pham Van Thang [b], Le Anh Vinh [c]

[a] *University of Science, Vietnam National University Hanoi, Viet Nam*

[b] *EPFL, Lausanne, Switzerland*

[c] *University of Education, Vietnam National University Hanoi, Viet Nam*

ARTICLE INFO

Article history:
Received 25 January 2016
Accepted 19 September 2016
Available online 17 October 2016

ABSTRACT

In this paper, we use methods from spectral graph theory to obtain some results on the sum–product problem over finite valuation rings $\mathcal{R}$ of order q^r which generalize recent results given by Hegyvári and Hennecart (2013). More precisely, we prove that, for related pairs of two-variable functions $f(x, y)$ and $g(x, y)$, if A and B are two sets in $\mathcal{R}^*$ with $|A| = |B| = q^\alpha$, then

$$\max\{|f(A, B)|, |g(A, B)|\} \gg |A|^{1+\Delta(\alpha)},$$

for some $\Delta(\alpha) > 0$.

1. Introduction

Let $\mathbb{F}_q$ be a finite field of q elements where q is an odd prime power. Throughout the paper q will be a large prime power. Let $\mathcal{A}$ be a non-empty subset of a finite field $\mathbb{F}_q$. We consider the sum set

$$\mathcal{A} + \mathcal{A} := \{a + b : a, b \in \mathcal{A}\}$$

and the product set

$$\mathcal{A} \cdot \mathcal{A} := \{a \cdot b : a, b \in \mathcal{A}\}.$$

Let $|\mathcal{A}|$ denote the cardinality of $\mathcal{A}$. Bourgain, Katz and Tao [6] showed that when $1 \ll |\mathcal{A}| \ll q$ then $\max(|\mathcal{A} + \mathcal{A}|, |\mathcal{A} \cdot \mathcal{A}|) \gg |\mathcal{A}|^{1+\epsilon}$, for some $\epsilon > 0$. This improves the trivial bound $\max\{|\mathcal{A} +$

E-mail addresses: hamlaoshi@gmail.com (L.Q. Ham), thang.pham@epfl.ch (P.V. Thang), vinhla@vnu.edu.vn (L.A. Vinh).

http://dx.doi.org/10.1016/j.ejc.2016.09.009

$\mathcal{A}|, |\mathcal{A}\cdot\mathcal{A}|\} \gg |\mathcal{A}|$. (Here, and throughout, $X \asymp Y$ means that there exist positive constants C_1 and C_2 such that $C_1Y < X < C_2Y$, and $X \ll Y$ means that there exists $C > 0$ such that $X \leq CY$). The precise statement of their result is as follows.

Theorem 1.1 (*Bourgain, Katz and Tao, [6]*). *Let $\mathcal{A}$ be a subset of $\mathbb{F}_q$ such that*

$$q^{\delta} < |\mathcal{A}| < q^{1-\delta}$$

for some $\delta > 0$. Then one has a bound of the form

$$\max\{|\mathcal{A}+\mathcal{A}|, |\mathcal{A}\cdot\mathcal{A}|\} \gg |\mathcal{A}|^{1+\epsilon}$$

for some $\epsilon = \epsilon(\delta) > 0$.

Note that the relationship between ϵ and δ in Theorem 1.1 is difficult to determine. In [14], Hart, Iosevich, and Solymosi obtained a bound that gives an explicit dependence of ϵ on δ. More precisely, if $|A+A| = m$ and $|A\cdot A| = n$, then

$$|A|^3 \leq \frac{cm^2n|A|}{q} + cq^{1/2}mn, \tag{1.1}$$

for some positive constant c. Inequality (1.1) implies a non-trivial sum–product estimate when $|A| \gg q^{1/2}$. Using methods from the spectral graph theory, the third listed author [27] improved (1.1) and as a result, obtained a better sum–product estimate.

Theorem 1.2 (*Vinh, [27]*). *For any set $A \subseteq \mathbb{F}_q$, if $|A+A| = m$, and $|A\cdot A| = n$, then*

$$|A|^2 \leq \frac{mn|A|}{q} + q^{1/2}\sqrt{mn}.$$

Corollary 1.3 (*Vinh, [27]*). *For any set $A \subseteq \mathbb{F}_q$, we have*
If $q^{1/2} \ll |A| \ll q^{2/3}$, then

$$\max\{|A+A|, |A\cdot A|\} \gg \frac{|A|^2}{q^{1/2}}.$$

If $|A| \gg q^{2/3}$, then

$$\max\{|A+A|, |A\cdot A|\} \gg (q|A|)^{1/2}.$$

It follows from Corollary 1.3 that if $|A| = p^{\alpha}$, then

$$\max\{|A+A|, |A\cdot A|\} \gg |A|^{1+\Delta(\alpha)},$$

where $\Delta(\alpha) = \min\{1 - 1/2\alpha, (1/\alpha - 1)/2\}$. In the case that q is a prime, Corollary 1.3 was proved by Garaev [11] using exponential sums. Cilleruelo [9] also proved related results using dense Sidon sets in finite groups involving $\mathbb{F}_q$ and $\mathbb{F}_q^* := \mathbb{F}_q \setminus \{0\}$ (see [9, Section 3] for more details).

We note that a variant of Corollary 1.3 was considered by Vu [29], and the statement is as follows.

Theorem 1.4 (*Vu, [29]*). *Let P be a non-degenerate polynomial of degree k in $\mathbb{F}_q[x,y]$. Then for any $A \subseteq \mathbb{F}_q$, we have*

$$\max\{|A+A|, |P(A)|\} \gtrsim \min\left\{|A|^{2/3}q^{1/3}, |A|^{3/2}q^{-1/4}\right\},$$

where we say that a polynomial P is non-degenerate if P cannot be presented as of the form $Q(L(x,y))$ with Q is a one-variable polynomial and L is a linear form in x and y.