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First order convergence of matroids[☆]

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ABSTRACT

The model theory based notion of the first order convergence unifies the notions of the left-convergence for dense structures and the Benjamini–Schramm convergence for sparse structures. It is known that every first order convergent sequence of graphs with bounded tree-depth can be represented by an analytic limit object called a limit modeling. We establish the matroid counterpart of this result: every first order convergent sequence of matroids with bounded branch-depth representable over a fixed finite field has a limit modeling, i.e., there exists an infinite matroid with the elements forming a probability space that has asymptotically the same first order properties. We show that neither of the bounded branch-depth assumption nor the representability assumption can be removed.

1. Introduction

The theory of combinatorial limits keeps attracting a growing amount of attention. Combinatorial limits have sparked many exciting developments in extremal combinatorics, in theoretical computer

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science, and other areas. Their significance is also evidenced by a recent monograph of Lovász [25]. The better understood case of convergence of dense structures originated in the series of papers by Borgs, Chayes, Lovász, Sós, Szegedy, and Vesztergombi [7,8,6,26,27] on the dense graph convergence, and the notion was applied in various settings including hypergraphs, partial orders, permutations, and tournaments [12,16,20–22,24]. The convergence of sparse structures (such as graphs with bounded maximum degree) referred to as the Benjamini–Schramm convergence [1,2,11,15] is less understood despite having links to many problems of high importance. For example, the conjecture of Aldous and Lyons [1] on Benjamini–Schramm convergent sequences of graphs is essentially equivalent to Gromov's question of whether all countable discrete groups are sofic. Other notions of convergence of sparse graphs were also proposed and studied [3–5,11,15].

In the light of many results on the convergence of graphs, one can ask whether a reasonable theory of matroid convergence can be developed. The first obstacle to building such a theory comes from the fact that matroids when viewed as hypergraphs (e.g. with edges being the bases) can be too sparse for the classical dense convergence approach to be directly applied, and too dense for the sparse convergence approach at the same time. For example, the number of bases of the graphic matroid of K_n is n^{n-2}, an exponentially small fraction of all $(n-1)$-element subsets of the edge set of K_n and an even tinier fraction of all subsets of the edge set, which rules out the dense convergence approach. On the other hand, each element of this matroid is contained in a non-constant number of bases, and it is impossible to follow the sparse convergence approach. We overcome this obstacle by adapting the notion of the first order convergence to matroids.

The notion of the first order convergence was introduced by Nešetřil and Ossona de Mendez [28,29] as an attempt to unify the convergence notions in the dense and sparse settings: a sequence of structures of a fixed type (e.g., graphs) is *first order convergent* if the probability that a random k-tuple of its elements has a first order property φ, converges for every choice of φ (a formal definition can be found in Section 2.4). It holds that every first order convergent sequence of dense structures is convergent in the dense sense and every first order convergent sequence of sparse structures is convergent in the Benjamini–Schramm sense.

In the analogy to graphons in the setting of dense graphs and graphings in the setting of sparse graphs, an analytic limit object called a *limit modeling* was proposed in [28,29] to represent asymptotic properties of first order convergent sequences. Unlike in the dense and sparse graph settings, it is not true that every first order convergent sequence of graphs has a limit modeling. For example, the sequence of Erdős–Rényi random graphs $G_{n,p}$ for $p \in (0, 1)$ is first order convergent with probability one but it has no limit modeling [29, Lemma 18]. In the same paper, Nešetřil and Ossona de Mendez showed the following.

Theorem 1.1. *Every first order convergent sequence of graphs with bounded tree-depth has a limit modeling.*

This result was extended to first convergent sequences of trees and graphs of bounded path-width [13,31]. Nešetřil and Ossona de Mendez [30] have recently shown that every first order convergent sequence of graphs from a nowhere-dense class of graphs has a limit modeling, which is the most general result possible for monotone classes of graphs [29, Theorem 25].

As a test that the approach to the matroid convergence based on the first order convergence is meaningful, it seems natural to prove the analogue of Theorem 1.1, which is actually one of our results (Theorem 1.2). On the way towards Theorem 1.2, we need to find a matroid parameter that can play the role of the graph tree-depth. We do so by introducing a parameter called branch-depth in Section 3. We believe that this matroid parameter is the right analogue of the graph tree-depth because it has the following properties, which we establish in this paper. We refer the reader to [32, Chapter 6] for a thorough discussion of the graph tree-depth.

- The branch-depth of a matroid corresponding to a graph G is at most the tree-depth of G.
- The branch-depth of a matroid corresponding to a graph G with tree-depth d is at least $\frac{1}{2}\log_2 d$ if G is 2-connected.
- The branch-depth is a minor monotone parameter (the same holds for graph tree-depth).
- The branch-depth of a matroid is at most the square of the length of its longest circuit (recall that the tree-depth of a graph G is at most the length of its longest path).