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An approach towards Schubert positivities of polynomials using Kraśkiewicz–Pragacz modules

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ABSTRACT

In this paper, we investigate properties of modules introduced by Kraśkiewicz and Pragacz which realize Schubert polynomials as their characters. In particular, we give some characterizations of modules having filtrations by Kraśkiewicz–Pragacz modules. In finding criteria for such filtrations, we calculate generating sets for the annihilator ideals of the lowest vectors in Kraśkiewicz–Pragacz modules and derive a projectivity result concerning Kraśkiewicz–Pragacz modules.

1. Introduction

Though Schubert polynomials originally arose from Schubert calculus on flag varieties, they also have purely algebro-combinatorial interests apart from the geometry of flag varieties. One of the possible methods for studying Schubert polynomials is via the modules introduced by Kraśkiewicz and Pragacz [5,6]. For each permutation w, Kraśkiewicz and Pragacz defined a certain representation $\mathcal{S}_w$ of the Lie algebra $\mathfrak{b}$ of all upper triangular matrices such that its character with respect to the subalgebra $\mathfrak{h}$ of all diagonal matrices is equal to the Schubert polynomial $\mathfrak{S}_w$ (precise definition of $\mathcal{S}_w$ will be given in Section 3). In this paper we call these modules *Kraśkiewicz–Pragacz modules* or *KP modules*.

Since Schubert polynomials are a kind of generalization of Schur functions, it is natural to ask for several analogues of properties of Schur functions for Schubert polynomials. One of our motivation

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for studying KP modules is the investigation of such properties, in particular, positivity properties for Schubert polynomials as generalizations of positivity properties of Schur functions. For example, it is a classical result that $\mathfrak{S}_u\mathfrak{S}_v$ is a positive sum of Schubert polynomials. This positivity property is classically proved using the geometry of flag varieties. On the other hand, in the case of Schur functions, the Schur-positivity of the product of two Schur functions can also be explained through representation theory: i.e. by considering irreducible decompositions of tensor-product modules over the Lie algebra $\mathfrak{gl}_n$. In this paper we make an approach towards similar explanation for the positivity of products of Schubert polynomials using KP modules: if we show that the tensor product module $\mathcal{S}_u \otimes \mathcal{S}_v$ has a filtration by KP modules (i.e. a filtration whose successive quotients are KP modules), then it gives a representation-theoretic proof for the positivity of product $\mathfrak{S}_u\mathfrak{S}_v$ (actually, the existence of such filtration has been now proved in a subsequent work [16] by the author using the results developed in this paper). Note that, contrary to the $\mathfrak{gl}_n$ case, $\mathfrak{b}$-modules in general have nontrivial extensions between them and thus we have to consider filtrations of modules, not only direct sum decompositions.

Another problem we want to pursue using KP modules is the positivity question for the "plethysm" of a Schur function with a Schubert polynomial. For a symmetric function s and a polynomial $f = x^\alpha + x^\beta + \cdots$, the plethysm of s and f is defined as $s[f] = s(x^\alpha, x^\beta, \ldots)$ (cf. [10, Section I.8]). The question is: is $s_\sigma[\mathfrak{S}_w]$ a positive sum of Schubert polynomials, for all partitions σ and permutations w? This positivity property of Schubert polynomials not known before comes naturally as a generalization of classically known Schur-positivity for plethysms of Schur functions. As before, this problem is related with a problem on the Schur-functor image $s_\sigma(\mathcal{S}_w)$ of a KP module: if this module has a filtration by KP modules then the desired Schubert-positivity of the plethysm follows (note that, in the subsequent work [16] just mentioned above, this problem has been also solved using the results from this paper, thus giving a positive answer to the plethysm question above). In this paper, motivated by such problems on Schubert polynomials and KP modules, we give some algebraic characterizations of modules having filtrations by KP modules.

KP modules are in some way similar to Demazure modules (of type A), the modules generated by an extremal vector in an irreducible representation of $\mathfrak{gl}_n$: they are both cyclic $\mathfrak{b}$-modules parametrized by the weight of the generators, and if the index permutation is 2143-avoiding then the KP module coincides with the Demazure module with the same weight of the generator (note that in general they are different (see Example 3.5): if a permutation w does not avoid 2143 then there exists a *strict* surjection from $\mathcal{S}_w$ to the Demazure module of corresponding lowest weight). In this paper, we develop a theory on KP modules in an analogous way as in the theory on Demazure modules [4,12,14], [15, Section 3].

The module $\mathcal{S}_w$ is generated by its lowest weight vector u_w. In this paper we first show in Section 4 that the annihilator ideal $\mathrm{Ann}_{\mathcal{U}(\mathfrak{n}^+)}(u_w)$, where $\mathfrak{n}^+$ is the Lie subalgebra of all strictly upper triangular matrices, is generated by the elements $e_{ij}^{m_{ij}(w)+1}$ $(1 \le i < j \le n)$ for some integers $m_{ij}(w)$ which can be read off from w, where e_{ij} denotes the (i,j)-th matrix unit. This result can be seen as a generalization of a classical result which states that the finite dimensional irreducible representation of $\mathfrak{gl}_n$ with lowest weight $-\lambda$ can be presented as $\mathcal{U}(\mathfrak{n}^+)/\langle e_i^{\langle \lambda, h_i \rangle + 1} \rangle_{1 \le i \le n-1}$ as a $\mathcal{U}(\mathfrak{n}^+)$-module. This result can moreover be seen as an analogue of the result on Demazure modules, given by Joseph [4, Theorem 3.4], which states, in the $\mathfrak{gl}_n$-case, that the annihilator of the generator of the Demazure module with lowest weight $\lambda \in \mathbb{Z}^n$ is generated by the elements $e_{ij}^{1+\max\{0,\lambda_j-\lambda_i\}}$ $(1 \le i < j \le n)$.

Using this presentation of KP modules, in Section 6 we characterize KP modules by their projectivity in certain categories; it is an analogue of Polo's theorem (originally for Demazure modules: see [12], [15, Section 3]) in the case of KP modules. Finally, using the results obtained so far, we obtain some criteria (Theorem 8.1, Theorem 8.2) for a module to have a filtration by KP modules, in a way similar to the argument given by van der Kallen [14], [15, Section 3] for Demazure modules using the method from the theory of highest-weight categories.

The paper is organized as follows. In Sections 2 and 3 we recall and define some basic notations and results about Schubert polynomials and KP modules. In Sections 4 and 5 we give a generating set for the annihilator ideal of the lowest weight vector in a KP module. In Section 6, we introduce new orderings on the weight lattice and show some results relating KP modules with these orderings. In Sections 7 and 8, we obtain some characterizations of modules having a filtration by KP modules using