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Cutting convex curves

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ABSTRACT

We show that for any two convex curves C_1 and C_2 in $\mathbb{R}^d$ parametrized by $[0, 1]$ with opposite orientations, there exists a hyperplane H with the following property: For any $t \in [0, 1]$ the points $C_1(t)$ and $C_2(t)$ are never in the same open half space bounded by H. This will be deduced from a more general result on equipartitions of ordered point sets by hyperplanes.

1. Introduction

In [3] the following interesting theorem is proved: *If $A_1, A_2, \dots, A_n$ and $B_1, B_2, \dots, B_n$ are the vertices of two convex polygons in the plane ordered cyclically with opposite orientation, then there exists a line that intersects each of the line segments A_jB_j.*

This result can be derived from a continuous version of the problem which has an elementary topological argument (which is what they do in [3]). The natural problem which is raised in [3] is to try to generalize this result to higher dimensions, and some partial results are proven for convex polytopes in $\mathbb{R}^3$ (but with some limitations).

Here we will give a generalization of this theorem to arbitrary dimensions. Our proof is essentially different from the one given in [3] and uses notions from oriented matroid theory together with a basic fixed-point theorem.

A *convex curve* in $\mathbb{R}^d$ is a continuous mapping $C\colon [0, 1] \to \mathbb{R}^d$ which intersects every hyperplane at most d times, meaning $|\{t \in [0, 1] : C(t) \in H\}| \le d$ for any hyperplane $H \subset \mathbb{R}^d$. The name comes

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from the fact that in $\mathbb{R}^2$ a convex curve corresponds to a connected subset of the boundary of a convex body. A typical example of a convex curve in $\mathbb{R}^d$ is the so-called *moment curve*,

$$\left\{\left(t, t^2, \ldots, t^d\right) : t \in [0, 1]\right\},$$

which has numerous applications in discrete and computational geometry. For instance, the convex hull of $n > d$ distinct points on the moment curve in $\mathbb{R}^d$ is a cyclic d-polytope [7], which is arguably the most useful example of a neighborly polytope.

An important feature of a convex curve in $\mathbb{R}^d$ is the fact that for any $0 \leq t_0 < t_1 < \cdots < t_d \leq 1$, the determinant

$$\det \begin{bmatrix} C(t_0) & C(t_1) & \cdots & C(t_d) \\ 1 & 1 & \cdots & 1 \end{bmatrix} \tag{1}$$

does not vanish, which is in fact a defining property of convex curves [5]. (In the case of a *closed* convex curve, e.g. $C(0) = C(1)$, we naturally require that $t_d < 1$.) This implies that the determinant (1) has the same sign for all choices $0 \leq t_0 < t_1 < \cdots < t_d \leq 1$, and therefore we may define the *orientation* of a convex curve C to be *positive* or *negative* according to the sign of the determinant (1).

In this note we report the following interesting property concerning *pairs* of convex curves.

Theorem 1.1. *Let C_1 and C_2 be convex curves in $\mathbb{R}^d$ with opposite orientations. There exists a hyperplane H such that the points $C_1(t)$ and $C_2(t)$ are never contained in the same open half space bounded by H.*

For $d = 2$ this is the main result shown in [3]. Somewhat surprisingly, the convexity plays a rather minor role. Theorem 1.1 will be deduced from a more general result concerning point sets, stated below as Theorem 2.1.

2. Order-types

Let A be a set of points in $\mathbb{R}^d$ which affinely spans $\mathbb{R}^d$. The *order-type* of A is the set of signs of the determinants

$$\det \begin{bmatrix} a_0 & a_1 & \cdots & a_d \\ 1 & 1 & \cdots & 1 \end{bmatrix} \tag{2}$$

indexed by the $(d+1)$-tuples $(a_0, a_1, \ldots, a_d) \in A^{d+1}$ with distinct entries. Notice that the condition that A affinely spans $\mathbb{R}^d$ guarantees the existence of at least one $(d+1)$-tuple such that the determinant (2) is non-zero. Usually, the notion of order-type is used with finite sets of points, however we will allow the possibility of A being infinite.

The order-type defines an equivalence relation on sets of points in $\mathbb{R}^d$, in which two sets A and B are equivalent if there exists a bijection $\gamma : A \to B$ with

$$\operatorname{sgn}\det \begin{bmatrix} a_0 & a_1 & \cdots & a_d \\ 1 & 1 & \cdots & 1 \end{bmatrix} = \operatorname{sgn}\det \begin{bmatrix} \gamma(a_0) & \gamma(a_1) & \cdots & \gamma(a_d) \\ 1 & 1 & \cdots & 1 \end{bmatrix} \tag{3}$$

for all $(d+1)$-tuples $(a_0, a_1, \ldots, a_d)$ with distinct entries (see e.g. [2]).

To the other extreme, we say that the sets A and B have *opposite* order-types if

$$\operatorname{sgn}\det \begin{bmatrix} a_0 & a_1 & \cdots & a_d \\ 1 & 1 & \cdots & 1 \end{bmatrix} = -\operatorname{sgn}\det \begin{bmatrix} \gamma(a_0) & \gamma(a_1) & \cdots & \gamma(a_d) \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

is satisfied instead of (3). In this case we say that γ is *order-type reversing*.

Theorem 2.1. *Let A and B be point sets in $\mathbb{R}^d$ which affinely span $\mathbb{R}^d$. If $\gamma : A \to B$ is an order-type reversing bijection, then there exists a hyperplane which intersects all the segments ab with $b = \gamma(a)$.*

Remark 2.2. The condition on the affine span of the point sets could be weakened, but this would involve refining the notion of the order-type (since all the determinants (2) would vanish) and the statement of Theorem 2.1 would become more technical.