European Journal of Combinatorics 58 (2016) 38–51

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Lattice path constructions for orthosymplectic determinantal formulas[☆]

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ARTICLE INFO

Article history:
Received 19 August 2015
Accepted 6 May 2016
Available online 4 June 2016

ABSTRACT

We give lattice path proofs of determinantal formulas for orthosymplectic characters. We use the $spo(2m, n)$-tableaux introduced by Benkart, Shader and Ram, which have both a semistandard symplectic part and a row-strict part. We obtain orthosymplectic Jacobi–Trudi identities and an orthosymplectic Giambelli identity by associating $spo(2m, n)$-tableaux to certain families of nonintersecting lattice paths and using an adaptation of the Gessel–Viennot method.

1. Introduction

Given a partition λ, the Jacobi–Trudi identity gives a determinantal expression for the Schur function $s_\lambda(x_1, \ldots, x_n)$ in terms of homogeneous symmetric polynomials and the dual Jacobi–Trudi identity gives an expression for $s_\lambda(x_1, \ldots, x_n)$ in terms of elementary symmetric functions (see [9]). The Giambelli identity gives a determinantal expression for $s_\lambda(x_1, \ldots, x_n)$ in terms of Schur functions associated to the principal hooks of the appropriate Young diagram of shape λ. Gessel and Viennot gave lattice path proofs of the Jacobi–Trudi identities by interpreting semistandard λ-tableaux as certain families of nonintersecting lattice paths [6] and Stembridge gave a lattice path proof of the Giambelli identity [11].

Semistandard symplectic tableaux index bases of irreducible representations of symplectic groups and symplectic Schur functions can be described in terms of the semistandard symplectic tableaux of King [8]. Fulmek and Krattenthaler [5] proved symplectic and orthogonal Jacobi–Trudi and Giambelli

☆ This research was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada (grant no. DDG-2015-00045).

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http://dx.doi.org/10.1016/j.ejc.2016.05.002
0195-6698/

identities for symplectic Schur functions and orthogonal Schur functions in [5] by interpreting semistandard symplectic tableaux as certain families of nonintersecting lattice paths.

Moving to the superalgebra setting, we have the supersymmetric Schur functions (or hook Schur functions), which are characters of irreducible representations of the Lie superalgebras $gl(m|n)$ [3]. Berele and Regev gave combinatorial descriptions of supersymmetric Schur functions in terms of hybrid Schur functions involving tableaux with both a column-strict part and a row-strict part. Goulden and Greene gave lattice path proofs of determinantal formulas for supersymmetric Schur functions in [7].

Balantekin and Bars gave Jacobi–Trudi formulas for characters of irreducible representations of orthosymplectic Lie algebras $spo(2m, n)$ in [1], which were proved using a different approach in [2]. Combinatorial descriptions of $spo(2m, n)$-characters were first given in [2] using $spo(2m, n)$-tableaux, which have both a symplectic part (in the sense of King [8]) and a row-strict part.

The aim of this paper is to give lattice path proofs of Jacobi–Trudi formulas and Giambelli formulas for characters of representations of orthosymplectic Lie superalgebras $spo(2m, n)$ using $spo(2m, n)$-tableaux.

In Section 3, we give lattice path proofs of the symplectic Jacobi–Trudi formulas and Giambelli formulas, using different families of paths than those used in [5] and some algebraic identities. Since orthosymplectic Schur functions can be described using tableaux that are hybrid semistandard symplectic tableaux and semistandard tableaux, our approach in Section 3 leads into that used in Section 4, where we give lattice path proofs of orthosymplectic determinantal identities.

2. Preliminaries

A *partition* is a k-tuple of integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. The *length* of λ is $\ell(\lambda) = k$. The *Young diagram* of shape λ consists of $N = \sum_{i=1}^{k} \lambda_i$ boxes in k left-justified rows with λ_i boxes in the ith row. The *conjugate* of λ is the partition $\lambda^t = (\lambda_1^t, \lambda_2^t, \ldots, \lambda_s^t)$ where λ_i^t denotes the number of boxes in the ith column of the Young diagram of shape λ. A λ-tableau is a filling of the Young diagram of shape λ with entries from a set $\{1, 2, \ldots, n\}$ of positive integers. A λ-tableau is *semistandard* if the entries in the rows are weakly increasing from left to right and the entries in the columns are strictly increasing from top to bottom.

Example 2.1. Let $\lambda = (3, 2, 2)$. The first λ-tableau below is semistandard, while the second is not.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 3 \\ \cline{1-2} 4 & 5 \\ \cline{1-2}\end{array}\,, \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 4 & 5 \\ \cline{1-2}\end{array}\,.$$

For a λ-tableau T, let $a_i(T)$ denote the number of entries equal to i in T. The *weight* of T is the monomial in the variables $X = \{x_1, x_2, \ldots, x_n\}$ defined by $\mathrm{wt}(T) = \prod_{i=1}^{n} x_i^{a_i(T)}$. The *Schur function* corresponding to λ is

$$s_\lambda(X) = \sum_T \mathrm{wt}(T),$$

where the sum runs over all semistandard λ-tableaux T with entries in $\{1, 2, \ldots, n\}$.

Define the rth *homogeneous symmetric function* by

$$h_r(X) = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq n} x_{i_1} \cdots x_{i_r},$$

and the rth *elementary symmetric function* by

$$e_r(X) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} x_{i_1} \cdots x_{i_r},$$

where both $h_r(X) = 0$ and $e_r(X) = 0$ if $r < 0$ and $h_0(X) = e_0(X) = 1$.