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Interlacing log-concavity of the derangement polynomials and the Eulerian polynomials

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ABSTRACT

Let $\mathbf{D}(n, k)$ be the set of derangements of $[n]$ with k excedances and $d(n, k)$ be the cardinality of $\mathbf{D}(n, k)$. We establish a bijection between $\mathbf{D}(n, k)$ and the set of labeled lattice paths of length n with k horizontal edges. Using this bijection, we give a direct combinatorial proof of the inequalities $d(n, k-1)d(m, l+1) < d(n, k)d(m, l)$ for $n \geq m \geq 1$ and $l \geq k \geq 1$. Moreover, we prove the interlacing log-concavity of the sequences $\{d(n, k)\}_{0 \leq k \leq n}$. By a similar combinatorial structure, we show that the Eulerian polynomials possess these properties.

1. Introduction

Unimodal and log-concave sequences and polynomials often arise in combinatorics, algebra and geometry, see Brenti [2,4,5], Stanley [9], and Stembridge [11]. A sequence $(a_n)_{n \geq 0}$ of real numbers is said to be log-concave if for $n \geq 1$,

$$a_n^2 \geq a_{n+1}a_{n-1}. \tag{1.1}$$

If the inequalities in (1.1) are strict, then $(a_n)_{n \geq 0}$ is said to be strictly log-concave. A polynomial is said to be log-concave if the sequence of its coefficients is log-concave.

For the sequences of $\{a(n, k)\}_{0 \leq k \leq n}$ $(n = 1, 2, \ldots)$, note that the log-concavity property only describes the connection among $a_{n,k}$ for fixed n. Chen, Wang and Xia [6] defined the interlacing log-concavity as follows. The sequence $\{a(n, k)\}_{0 \leq k \leq n}$ $(n = 1, 2, \ldots)$ is said to be interlacingly log-concave

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if for $m \geq 1,\ k \geq 1$,

$$\frac{a_k(m+1)}{a_{k+1}(m+1)} < \frac{a_k(m)}{a_{k+1}(m)} < \frac{a_{k+1}(m+1)}{a_{k+2}(m+1)}. \tag{1.2}$$

Obviously, the interlacingly log-concave property implies the log-concavity. Chen, Wang and Xia [6] showed the interlacing log-concavity of the Boros–Moll polynomials. They also gave a sufficient condition for interlacing log-concavity of a sequence of polynomials. Applying this criterion, they proved that the Stirling numbers of the first kind without signs, the Stirling numbers of the second kind and the Whitney numbers are interlacingly log-concave. Thus, we consider whether other combinatorial polynomials process this property. In this paper, we shall prove that the derangement polynomials and the Eulerian polynomials are interlacingly log-concave. Since the criterion in [6] does not work for these two polynomials, we will prove these results by a direct combinatorial injection.

Recall that $\mathfrak{S}_n$ denotes the symmetric group on $[n] = \{1, 2, \ldots, n\}$. Let π be a permutation in $\mathfrak{S}_n$. An element $i \in [n]$ is called a weak excedance of π if $\pi(i) \geq i$, while i is called an excedance of π if $\pi(i) > i$. We say that i is a descent (ascent, respectively) of π if $\pi_i > \pi_{i+1}$ ($\pi_i < \pi_{i+1}$, respectively).

A permutation π is called a derangement if $\pi(i) \neq i$ for all $i \in [n]$. Let D_n denote the set of all derangements of $\mathfrak{S}_n$. The derangement polynomial is defined by Brenti as

$$d_n(q) = \sum_{\pi \in D_n} q^{exc(\pi)}, \tag{1.3}$$

where $\mathrm{exc}(\pi)$ is the number of excedances of π. The coefficient of q^k in $d_n(q)$ is denoted by $d(n, k)$.

Brenti [3] showed that for $n \geq 1$, the polynomials $d_n(q)$ are symmetric and unimodal. Furthermore, He conjectured that $d_n(q)$ has only real zeros for $n \geq 1$. In 2007, this conjecture was independently proved by Liu and Wang [7] and Canfield (unpublished, see Remark 3.2 in [7]). This leads to an algebraic proof of the log-concave property of $\{d(n, k)\}_{0 \leq k \leq n}$. So far, no combinatorial proof is known of the log-concavity of $\{d(n, k)\}_{0 \leq k \leq n}$.

The main objective of this paper is to prove the interlacing log-concavity of the derangement polynomials. In Section 2, we first give a bijection between the derangements and the labeled lattice paths. Based on this bijection, we will give a direct combinatorial proof of the log-concavity of the derangement polynomials. Moreover, we prove that the derangement polynomials are interlacingly log-concave. As a conclusion, we show the interlacing log-concavity of Eulerian polynomials by a similar combinatorial structure.

2. Labeled lattice paths and derangements

In this section, we establish a bijection between the set of derangements of $[n]$ with k excedances and the set of labeled lattice paths of length n with k horizontal steps.

Recall that the Foata's First Fundamental Transformation is a bijection of the symmetric group onto itself, which can be described as follows. Given a permutation π, consider its decomposition as a product of cycles. Then write each cycle with its largest element and arrange the cycles in increasing order of their first elements. Finally, erase the parentheses. Let $\hat{\pi}$ be the resulting permutation. For example, given $\pi = 4\,271\,365$ with its cycle notation $(14)(2)(375)(6)$, we have $\hat{\pi} = 2\,416\,753$.

Proposition 2.1 ([10]). *The bijection $f : \pi \to \hat{\pi}$ maps the set of n-permutations with k weak excedances onto the set of n-permutations with $k-1$ ascents.*

Proof. Assume $\pi = (a_1 a_2 \cdots a_{i_1})(a_{i_1+1} a_{i_1+2} \cdots a_{i_2}) \cdots (a_{i_{k-1}+1} a_{i_{k-1}+2} \cdots a_{i_k})$ is a permutation in standard form and has k weak excedances. Then $a_1 < a_{i_1+1} < \cdots < a_{i_{k-1}+1}$ and $a_1, a_{i_1+1}, \ldots, a_{i_{k-1}+1}$ are respectively the largest elements of their cycles. It is easy to show that $a_j < a_{j+1}$ in π if and only if $f(a_j) > a_j$. This implies that the number of descents of $\hat{\pi}$ is equal to the number of non-weak excedances of π. If the number of non-weak excedances of π is k, then the number of descents of $\hat{\pi}$ is $n-k$. It follows that $\hat{\pi}$ has $k-1$ ascents. This completes the proof. ■