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Multiply union families in $\mathbb{N}^n$

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A R T I C L E I N F O

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A B S T R A C T

Let $A \subset \mathbb{N}^n$ be an r-wise s-union family, that is, a family of sequences with n components of non-negative integers such that for any r sequences in A the total sum of the maximum of each component in those sequences is at most s. We determine the maximum size of A and its unique extremal configuration provided (i) n is sufficiently large for fixed r and s, or (ii) $n = r + 1$.

1. Introduction

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$ denote the set of non-negative integers, and let $[n] := \{1, 2, \ldots, n\}$. Intersecting families in $2^{[n]}$ or $\{0, 1\}^n$ are one of the main objects in extremal set theory. The equivalent dual form of an intersecting family is a union family, which is the subject of this paper. In [5] Frankl and Tokushige proposed to consider such problems not only in $\{0, 1\}^n$ but also in $[q]^n$. They determined the maximum size of 2-wise s-union families (i) in $[q]^n$ for $n > n_0(q, s)$, and (ii) in $\mathbb{N}^3$ for all s (the definitions will be given shortly). In this paper we extend their results and determine the maximum size and structure of r-wise s-union families in $\mathbb{N}^n$ for the following two cases: (i) $n \geq n_0(r, s)$, and (ii) $n = r + 1$. Much research has been done for the case of families in $\{0, 1\}^n$, and there are many challenging open problems. The interested reader is referred to [2–4,8,9].

For a vector $\mathbf{x} \in \mathbb{R}^n$, we write x_i or $(\mathbf{x})_i$ for the ith component, so $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Define the *weight* of $\mathbf{a} \in \mathbb{N}^n$ by

$$|\mathbf{a}| := \sum_{i=1}^{n} a_i.$$

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For a finite number of vectors $\mathbf{a}, \mathbf{b}, \dots, \mathbf{z} \in \mathbb{N}^n$ define the join $\mathbf{a} \vee \mathbf{b} \vee \cdots \vee \mathbf{z}$ by

$$(\mathbf{a} \vee \mathbf{b} \vee \cdots \vee \mathbf{z})_i := \max\{a_i, b_i, \dots, z_i\},$$

and we say that $A \subset \mathbb{N}^n$ is r-wise *s-union* if

$$|\mathbf{a}_1 \vee \mathbf{a}_2 \vee \cdots \vee \mathbf{a}_r| \le s \quad \text{for all } \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_r \in A.$$

In this paper we address the following problem.

Problem 1. For given n, r and s, determine the maximum size $|A|$ of r-wise s-union families $A \subset \mathbb{N}^n$.

To describe candidates A that give the maximum size to the above problem, we need some more definitions. Let us introduce a partial order $\prec$ in $\mathbb{R}^n$. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ we let $\mathbf{a} \prec \mathbf{b}$ iff $a_i \le b_i$ for all $1 \le i \le n$. Then we define a *down set* for $\mathbf{a} \in \mathbb{N}^n$ by

$$\mathcal{D}(\mathbf{a}) := \{\mathbf{c} \in \mathbb{N}^n : \mathbf{c} \prec \mathbf{a}\},$$

and for $A \subset \mathbb{N}^n$ let

$$\mathcal{D}(A) := \bigcup_{\mathbf{a} \in A} \mathcal{D}(\mathbf{a}).$$

We also introduce $\mathcal{S}(\mathbf{a}, d)$, which can be viewed as a part of sphere centered at $\mathbf{a} \in \mathbb{N}^n$ with radius $d \in \mathbb{N}$, defined by

$$\mathcal{S}(\mathbf{a}, d) := \{\mathbf{a} + \boldsymbol{\epsilon} \in \mathbb{N}^n : \boldsymbol{\epsilon} \in \mathbb{N}^n, \ |\boldsymbol{\epsilon}| = d\}.$$

We say that $\mathbf{a} \in \mathbb{N}^n$ is a *balanced partition*, if all a_i's are as close to each other as possible, more precisely, $|a_i - a_j| \le 1$ for all i, j. Let $\mathbf{1} := (1, 1, \dots, 1) \in \mathbb{N}^n$.

For $r, s, n, d \in \mathbb{N}$ with $0 \le d \le \lfloor \frac{s}{r} \rfloor$ and $\mathbf{a} \in \mathbb{N}^n$ with $|\mathbf{a}| = s - rd$ let us define a family K by

$$K = K(r, n, \mathbf{a}, d) := \bigcup_{i=0}^{\lfloor \frac{d}{u} \rfloor} \mathcal{D}(\mathcal{S}(\mathbf{a} + i\mathbf{1}, d - ui)), \tag{1}$$

where $u = n - r + 1$. This is the candidate family. Intuitively K is a union of balls, and the corresponding centers and radii are chosen so that K is r-wise s-union as we will see in Claim 3 in the next section.

Conjecture 1. *Let $r \ge 2$ and s be positive integers. If $A \subset \mathbb{N}^n$ is r-wise s-union, then*

$$|A| \le \max_{0 \le d \le \lfloor \frac{s}{r} \rfloor} |K(r, n, \mathbf{a}, d)|,$$

where $\mathbf{a} \in \mathbb{N}^n$ is a balanced partition with $|\mathbf{a}| = s - rd$. Moreover if equality holds, then $A = K(r, n, \mathbf{a}, d)$ for some $0 \le d \le \lfloor \frac{s}{r} \rfloor$.

We first verify the conjecture when n is sufficiently large for fixed r, s. Let $\mathbf{e}_i$ be the ith standard base of $\mathbb{R}^n$, that is, $(\mathbf{e}_i)_j = \delta_{ij}$. Let $\tilde{\mathbf{e}}_0 = \mathbf{0}$, and $\tilde{\mathbf{e}}_i = \sum_{j=1}^{i} \mathbf{e}_j$ for $1 \le i \le n$, e.g., $\tilde{\mathbf{e}}_n = \mathbf{1}$.

Theorem 1. *Let $r \ge 2$ and s be fixed positive integers. Write $s = dr + p$ where d and p are non-negative integers with $0 \le p < r$. Then there exists an $n_0(r, s)$ such that if $n > n_0(r, s)$ and $A \subset \mathbb{N}^n$ is r-wise s-union, then*

$$|A| \le \left|\mathcal{D}(\mathcal{S}(\tilde{\mathbf{e}}_p, d))\right|.$$

Moreover if equality holds, then A is isomorphic to $\mathcal{D}(\mathcal{S}(\tilde{\mathbf{e}}_p, d)) = K(r, n, \tilde{\mathbf{e}}_p, d)$.

We mention that the case $A \subset \{0, 1\}^n$ of Conjecture is posed in [2] and partially solved in [2,3], and the case $r = 2$ of Theorem 1 is proved in [5] in a slightly stronger form. We also notice that