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Eventual quasi-linearity of the Minkowski length

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ABSTRACT

The Minkowski length of a lattice polytope P is a natural generalization of the lattice diameter of P. It can be defined as the largest number of lattice segments whose Minkowski sum is contained in P. The famous Ehrhart theorem states that the number of lattice points in the positive integer dilates tP of a lattice polytope P behaves polynomially in $t \in \mathbb{N}$. In this paper we prove that for any lattice polytope P, the Minkowski length of tP for $t \in \mathbb{N}$ is eventually a quasi-polynomial with linear constituents. We also give a formula for the Minkowski length of coordinates boxes, degree one polytopes, and dilates of unimodular simplices. In addition, we give a new bound for the Minkowski length of lattice polygons and show that the Minkowski length of a lattice triangle coincides with its lattice diameter.

0. Introduction

Let P be a d-dimensional lattice polytope in $\mathbb{R}^d$. Recall that the lattice diameter $\ell(P)$ is defined as one less than the largest number of collinear lattice points in P. The Minkowski length is a natural extension of this notion. For any $1 \leq n \leq d$, let $L_n(P)$ be the largest number of lattice polytopes of positive dimension whose Minkowski sum is at most n-dimensional and is contained in P. We call $L_n(P)$ the nth Minkowski length of P, and $L(P) = L_d(P)$ simply the Minkowski length of P. Note $L_1(P)$ coincides with the lattice diameter $\ell(P)$, as in this case the Minkowski summands are collinear

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lattice segments. It is not hard to show (see the discussion after Definition 1.1) that $L_n(P)$ is the largest number of lattice segments whose Minkowski sum is at most n-dimensional and is contained in P.

The Minkowski length $L(P)$ of a lattice polytope $P \subset \mathbb{R}^d$ was first introduced in [7] in relation to studying parameters of toric surface codes. Every lattice polytope P defines a space $\mathcal{L}(P)$ of Laurent polynomials (over some field) whose monomials have exponent vectors lying in P. Such spaces naturally appear in the theory of toric varieties. The algebraic interpretation of the Minkowski length is the following: $L(P)$ is the largest number of irreducible factors a polynomial $f \in \mathcal{L}(P)$ may have. This information is particularly important when one studies zeros of polynomials in $\mathcal{L}(P)$, see [6,5,7,8,4]. A number of results concerning $L(P)$ appeared in [3,7,9].

Let $tP = \{tx \in \mathbb{R}^d \mid x \in P\}$ be the dilate of P by a positive integer factor t. The main result of this paper explains the behavior of $L_n(tP)$ as a function of the scaling factor $t \in \mathbb{N}$ in the spirit of the Ehrhart theory. In Theorem 2.20 we prove that for any lattice polytope P the function $L_n(tP)$ is eventually quasi-polynomial with linear constituents (we say "quasi-linear" for short), which contributes positively to the "ubiquitousness of quasi-polynomials" phenomenon declared by Kevin Woods [10]. For an introduction to the Ehrhart theory we refer the reader to the wonderful book by M. Beck and S. Robins [2].

To prove eventual quasi-linearity of the Minkowski lengths $L_n(P)$ we define and study their rational counterparts: a sequence of rational numbers $\lambda_1(P) \leq \cdots \leq \lambda_d(P) = \lambda(P)$ associated with P. Here $\lambda_1(P)$ is the rational diameter of P and $\lambda_n(P)$ is the "asymptotic" Minkowski length, i.e. $\lambda_n(P) = \lim_{t\to\infty} L_n(tP)/t$. In Theorem 2.15 we prove that $\lambda_n(P) = L_n(kP)/k$ for some $k \in \mathbb{N}$.

Although an algorithm for computing $L(P)$ was presented in dimensions two and three (see [3,7]), there have been no explicit formulas for $L(P)$ even for simplices. Here we prove that $L(t\Delta) = t$ for any unimodular simplex Δ and any $t \in \mathbb{N}$. This result allowed us to write explicit answers for $L(P)$ for other classes of polytopes such as coordinate boxes and polytopes of degree one (see Corollary 2.2 and examples afterwards). In Section 3 we prove that for lattice triangles the Minkowski length coincides with the lattice diameter. The final part of the paper contains some examples and open questions.

1. Preliminaries

We start with some standard terminology from geometric combinatorics. A polytope $P \subset \mathbb{R}^d$ is called *lattice* (resp. *rational*) if its vertices have integer (resp. rational) coordinates. A vector $v \in \mathbb{Z}^d$ is called *primitive* if the greatest common divisor of its coordinates is 1. A lattice segment is called *primitive* if it contains exactly two lattice points. A d-dimensional simplex is called *unimodular* if its vertices affinely generate the lattice $\mathbb{Z}^d$.

Given a lattice polytope $P \subset \mathbb{R}^d$, denote by $\mathrm{Vol}_d(P)$ the Euclidean d-dimensional volume of P. Note that the d-dimensional volume of any parallelepiped formed by a basis of $\mathbb{Z}^d$ equals 1. More generally, suppose P is contained in an n-dimensional rational affine subspace $a+H$ for a rational linear subspace $H \subset \mathbb{R}^d$ and $a \in \mathbb{Q}^d$. Denote by $\mathrm{Vol}_n(P)$ the n-dimensional volume of P normalized such that the n-dimensional volume of any parallelepiped formed by a basis of the lattice $H \cap \mathbb{Z}^d$ equals 1.

1.1. Minkowski length

Let P and Q be convex polytopes in $\mathbb{R}^d$. Their *Minkowski sum* is the set

$$P + Q = \{p + q \in \mathbb{R}^d \mid p \in P,\ q \in Q\},$$

which is again a convex polytope.

Definition 1.1. Let P be a lattice polytope in $\mathbb{R}^d$. Define the *Minkowski length* $L = L(P)$ of P to be the largest number of lattice polytopes $Q_1, \dots, Q_L$ of positive dimension whose Minkowski sum is contained in P. Any such sum $Q_1 + \cdots + Q_L$ is called a *maximal decomposition in P*.

We refer the reader to [3] for examples illustrating this definition. It is clear from the definition that $L(P)$ is monotone with respect to inclusion: $L(P) \leq L(Q)$ if $P \subseteq Q$, and is superadditive with