European Journal of Combinatorics 58 (2016) 144–165

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Tableau sequences, open diagrams, and Baxter families

Sophie Burrill [a], Julien Courtiel [a], Eric Fusy [b,c], Stephen Melczer [d,e], Marni Mishna [a]

[a] *Department of Mathematics, Simon Fraser University, Burnaby, Canada*
[b] *PIMS (CNRS UMI 3069), University of British Columbia, Vancouver, Canada*
[c] *LIX, École Polytechnique, Palaiseau, France*
[d] *Cheriton School of Computer Science, University of Waterloo, Waterloo, ON, Canada*
[e] *U. Lyon, CNRS, ENS de Lyon, Inria, UCBL, Laboratoire LIP, France*

A R T I C L E   I N F O

Article history:
Received 11 June 2015
Accepted 26 May 2016
Available online 23 June 2016

A B S T R A C T

Walks on Young's lattice of integer partitions encode many objects of algebraic and combinatorial interest. Chen et al. established connections between such walks and arc diagrams. We show that walks that start at $\varnothing$, end at a row shape, and only visit partitions of bounded height are in bijection with a new type of arc diagram — open diagrams. Remarkably, two subclasses of open diagrams are equinumerous with well known objects: standard Young tableaux of bounded height, and Baxter permutations. We give an explicit combinatorial bijection in the former case, and a generating function proof and new conjecture in the second case.

1. Introduction

The lattice of partition diagrams, where domination is given by inclusion of Ferrers diagrams, is known as Young's lattice. Walks on this lattice are important since they encode many objects of combinatorial and algebraic interest. A walk on Young's lattice can be listed as a sequence of Ferrers diagrams such that at most a single box is added or deleted at each step. A class of such sequences is

E-mail addresses: srb7@sfu.ca (S. Burrill), jcourtie@math.ubc.ca (J. Courtiel), fusy@lix.polytechnique.fr (E. Fusy), smelczer@uwaterloo.ca (S. Melczer), mmishna@sfu.ca (M. Mishna).

http://dx.doi.org/10.1016/j.ejc.2016.05.011
0195-6698/

also known as a *tableau family*. It is well known that there are several combinatorial classes in explicit bijection with tableau families ending in an empty shape, in particular when there are restrictions on the height of the tableaux which appear.

In this work we launch the study of tableau families that start at the empty partition and end with a partition composed of a single part: $\lambda = (m),\ m \geq 0$. Additionally, they are bounded, meaning that they only visit partitions that have at most k parts, for some fixed k. Remarkably, these have direct connections to both Young tableaux of bounded height and Baxter permutations. More precisely, we adapt results of Chen et al. [14] to the open diagrams of Burrill et al. [12], and use generating results of Bousquet-Mélou and Xin [9] to give proofs that these two classic combinatorial classes are in bijection with bounded height tableau families.

1.1. Part 1. Oscillating tableaux and Young tableaux of bounded height

The first tableau family that we consider is the set of *oscillating tableaux* with height bounded by k. These appear in the proofs of results on partitions avoiding certain nesting and crossing patterns [14], although they have a much longer history. They appear in the representation theory of the symplectic group, and elsewhere as up–down tableaux [6,35]. Our first main result is a new bijection connecting oscillating tableaux to the class of standard Young tableau of bounded height. Young tableaux are more commonly associated with oscillating tableau with no deletion step, but ours is a very different bijection. This result demonstrates a new facet of the ubiquity of Young tableaux.

Theorem 1. *The set of oscillating tableaux of size n with height bounded by k, which start at the empty partition and end in a row shape $\lambda = (m)$, is in bijection with the set of standard Young tableaux of size n with height bounded by $2k$, with m odd columns.*

The proof of Theorem 1 is by an explicit bijection between the two classes, an example of which is illustrated in Fig. 1. A slightly less refined version of this theorem was conjectured in an extended abstract version of this work [13]. Independently, but simultaneously to our own work, Krattenthaler [26] determined a different bijective map. Notably, he gave the interpretation of the m parameter as the number of odd columns.

One consequence of the bijective map is the symmetric joint distribution of two kinds of nesting patterns inside the class of involutions. Enumerative formulas for Young tableaux of bounded height have been known for almost half a century [20,21,5], but new generating function results can be derived from Theorem 1, notably an expression which can be written as a diagonal of a multivariate rational function. The analytic consequences of Theorem 1 are the subject of Section 4.3.

1.2. Part 2. Hesitating tableaux and Baxter permutations

In the second part, we consider the family of *hesitating tableaux*. These tableau sequences appear in studies of set partitions avoiding so-called *enhanced* nesting and crossing patterns. We make a generating function argument to connect hesitating tableaux that end in a row shape to Baxter permutations. A first computational proof of this identity was produced by Xin and Zhang [36]. Here we offer a slight variation on the computation and provide the intermediary details, using formulas of Bousquet-Mélou and Xin [9]. Specifically, the result is the following.

Theorem 2. *The number of hesitating tableaux of length $2n$ of height at most two and ending in a row is equal to the number B_{n+1} of Baxter permutations of length $n+1$, where*

$$B_n = \sum_{k=1}^{n} \frac{\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}}{\binom{n+1}{1}\binom{n+1}{2}}. \tag{1}$$

This theorem is a good candidate for a combinatorial proof. Baxter numbers have been described as the “big brother” of the well known Catalan numbers: they are the counting series for many