European Journal of Combinatorics 58 (2016) 203–224

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

The price of connectivity for cycle transversals[☆]

Tatiana R. Hartinger[a], Matthew Johnson[b], Martin Milanič[a], Daniël Paulusma[b]

[a] University of Primorska, UP IAM and UP FAMNIT, Koper, Slovenia

[b] School of Engineering and Computing Sciences, Durham University, UK

A R T I C L E   I N F O

Article history:
Received 12 October 2015
Accepted 5 June 2016
Available online 29 June 2016

A B S T R A C T

For a family of graphs $\mathcal{F}$, an $\mathcal{F}$-transversal of a graph G is a subset $S \subseteq V(G)$ that intersects every subset of $V(G)$ that induces a subgraph isomorphic to a graph in $\mathcal{F}$. Let $t_{\mathcal{F}}(G)$ be the minimum size of an $\mathcal{F}$-transversal of G, and $ct_{\mathcal{F}}(G)$ be the minimum size of an $\mathcal{F}$-transversal of G that induces a connected graph. For a class of connected graphs $\mathcal{G}$, we say that the price of connectivity of $\mathcal{F}$-transversals is multiplicative if, for all $G \in \mathcal{G}$, $ct_{\mathcal{F}}(G)/t_{\mathcal{F}}(G)$ is bounded by a constant, and additive if $ct_{\mathcal{F}}(G) - t_{\mathcal{F}}(G)$ is bounded by a constant. The price of connectivity is identical if $t_{\mathcal{F}}(G)$ and $ct_{\mathcal{F}}(G)$ are always equal and unbounded if $ct_{\mathcal{F}}(G)$ cannot be bounded in terms of $t_{\mathcal{F}}(G)$. We study classes of graphs characterized by one forbidden induced subgraph H and $\mathcal{F}$-transversals where $\mathcal{F}$ contains an infinite number of cycles and, possibly, also one or more anticycles or short paths. We determine exactly those classes of connected H-free graphs where the price of connectivity of these $\mathcal{F}$-transversals is unbounded, multiplicative, additive, or identical. In particular, our tetrachotomies extend known results for the case when $\mathcal{F}$ is the family of all cycles.

☆ The research in this paper was supported by a London Mathematical Society Scheme 4 Grant and by EPSRC Grant EP/K025090/1, and, in part, by the Slovenian Research Agency (I0-0035, research programs P1-0285, research projects N1-0032, J1-5433, J1-6720, J1-6743, J1-7051, and a Young Researchers Grant). An extended abstract of the paper appeared in the proceedings of MFCS 2015 [19].

E-mail addresses: tatiana.hartinger@iam.upr.si (T.R. Hartinger), matthew.johnson2@durham.ac.uk (M. Johnson), martin.milanic@upr.si (M. Milanič), daniel.paulusma@durham.ac.uk (D. Paulusma).

http://dx.doi.org/10.1016/j.ejc.2016.06.003

1. Introduction

Let $\mathcal{F}$ be a family of graphs. A graph is *$\mathcal{F}$-free* if it contains no induced subgraph isomorphic to some graph in $\mathcal{F}$ (if $\mathcal{F} = \{F\}$ for some graph F then we write F-free instead). An *$\mathcal{F}$-transversal* of a graph $G = (V, E)$ is a subset $S \subseteq V$ such that $G - S$ is $\mathcal{F}$-free; that is, S intersects every subset of V that induces a subgraph isomorphic to a graph in $\mathcal{F}$. In certain cases, $\mathcal{F}$-transversals are well studied. For example, a *vertex cover* is a $\{P_2\}$-transversal (here, P_k is the path on k vertices). Note that, for any $\{P_2\}$-transversal S of a graph G, the graph $G - S$ is an independent set. To give another example, a *feedback vertex set* is an $\mathcal{F}$-transversal for the infinite family $\mathcal{F} = \{C_3, C_4, C_5, \ldots\}$ (where C_k is the cycle on k vertices). In this case, for any $\mathcal{F}$-transversal S of a graph G, the graph $G - S$ is a forest. As the examples suggest, it is natural to study minimum size $\mathcal{F}$-transversals.

We can put an additional constraint on an $\mathcal{F}$-transversal S of a connected graph G by requiring that the subgraph of G induced by S is connected. Minimum size *connected* $\mathcal{F}$-transversals of a graph have also been investigated. In particular, minimum size connected vertex covers are well studied (see, for example, [4,6,8,11,14,17,21,23]) and minimum size connected feedback vertex sets have also received attention (see, for example, [2,10,18,20,22]). We study the following question:

> *What is the effect of adding the connectivity constraint on the minimum size of an $\mathcal{F}$-transversal for a graph family $\mathcal{F}$?*

We first give two definitions: for a connected graph G, let $t_{\mathcal{F}}(G)$ denote the minimum size of an $\mathcal{F}$-transversal of G, and let $ct_{\mathcal{F}}(G)$ denote the minimum size of a connected $\mathcal{F}$-transversal of G. So our aim is to find relationships between $ct_{\mathcal{F}}(G)$ and $t_{\mathcal{F}}(G)$; more particularly, we ask for a class of connected graphs $\mathcal{G}$, whether we can find a bound for $ct_{\mathcal{F}}(G)$ in terms of $t_{\mathcal{F}}(G)$ that holds for all $G \in \mathcal{G}$.

We briefly survey existing work starting with a number of results on vertex cover, that is, for $\mathcal{F} = \{P_2\}$. Cardinal and Levy [8] proved that for every $\epsilon > 0$ there is a multiplicative bound of $2/(1 + \epsilon) + o(1)$ in the class of connected n-vertex graphs with average degree at least ϵn; that is, $ct_{\mathcal{F}}(G) \le (2/(1 + \epsilon) + o(1))t_{\mathcal{F}}(G)$ for such graphs G. Camby et al. [6] proved that for the class of all connected graphs, there is a multiplicative bound of 2 and that this bound is asymptotically sharp for paths and cycles. They also gave forbidden induced subgraph characterizations of classes of graphs such that for every connected induced subgraph there is a multiplicative bound of t, for each $t \in \{1, 4/3, 3/2\}$.

Belmonte et al. [2,3] studied feedback vertex sets, that is, $\mathcal{F}$-transversals where $\mathcal{F} = \{C_3, C_4, C_5 \ldots\}$. They determined all finite families of graphs $\mathcal{H}$ such that for all connected graphs G in the class of $\mathcal{H}$-free graphs, $ct_{\mathcal{F}}(G)/t_{\mathcal{F}}(G)$ is bounded by a constant [3]. They also determined exactly those graphs classes $\mathcal{G}$ of H-free graphs for which, for all connected $G \in \mathcal{G}$, $ct_{\mathcal{F}}(G) - t_{\mathcal{F}}(G)$ is bounded by a constant (and they found exactly when that constant is zero) [2].

We also give two other examples of graph properties where the effect of requiring connectivity has been studied. A result of Duchet and Meyniel [13] implies that for all connected graphs the minimum size of a connected dominating set is at most 3 times the size of a minimum size dominating set. A result of Zverovich [24] implies that for connected (P_5, C_5)-free graphs this bound is exactly 1. Camby and Schaudt [7] showed that the equivalent multiplicative bound for connected (P_8, C_8)-free graphs is 2 and for connected (P_9, C_9)-free graphs it is 3; both bounds were shown to be sharp. They also proved that the problem of deciding whether, for a given class of graphs this bound is at most r is $P^{\mathrm{NP[log]}}$-complete for every fixed rational r with $1 < r < 3$. The same authors also found an example of an additive bound: they proved that for every connected (P_6, C_6)-free graph, a minimum size connected dominating set contains at most one more vertex than a minimum size dominating set. Grigoriev and Sitters [18] proved that for connected planar graphs of minimum degree at least 3, a minimum size connected face hitting set is at most 11 times larger than a minimum size face hitting set. Schweitzer and Schweitzer [22] reduced this bound to 5 and proved tightness.

In this paper we consider a number of families $\mathcal{F}$ that contain cycles, paths and complements of cycles. We study $\mathcal{F}$-transversals for graph classes characterized by one forbidden induced subgraph and ask whether the size of a minimum size *connected* $\mathcal{F}$-transversal can be bounded in terms of the size of a minimum size $\mathcal{F}$-transversal. Before we can present our results we need to introduce some additional terminology and notation.