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Monochromatic paths for the integers

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ABSTRACT

Recall that van der Waerden's theorem states that any finite coloring of the naturals has arbitrarily long monochromatic arithmetic sequences. We explore questions about the set of differences of those sequences.

1. Introduction

A famous result of van der Waerden asserts that any finite coloring of the positive integers contains arbitrarily long monochromatic arithmetic progressions. In a typical *Ramsey problem*, we try to prove that some particular type of structure cannot be avoided. In this article we will consider arithmetic progressions with common difference in some given set L, and sets whose consecutive differences belong to a fixed set.

We call a set $L \subset \mathbb{N}$ an r-*ladder* if for any r-coloring $\alpha : \mathbb{N} \to [1, r]$ there are arbitrarily long monochromatic arithmetic progressions with common difference $r \in L$. If a set $L \subseteq \mathbb{N}$ is an r-ladder for all positive integers r, then it is called a *ladder*. The set of positive integers $\mathbb{N}$ is an example of a ladder. The following related important open question has been proposed in [2].

Problem 1 (*Open*)**.** Is every 2-ladder set necessarily an r-ladder for every $r > 2$?

It is not even known if the r-ladder property implies $(r + 1)$-ladder property for any $r \geq 2$.

The following regularity property appears also in [2].

Theorem 1. *If $L = L_1 \cup L_2$ is a ladder, then either L_1 or L_2 is a ladder. If L is a ladder, then $L \cap n\mathbb{N}$ is a ladder for any $n \in \mathbb{N}$.*

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Proof. To prove the first statement, suppose neither L_1 nor L_2 is a ladder. Then there is a finite coloring of the positive integers for which there are no monochromatic arithmetic progressions with say k elements with difference in L_1, and a similar coloring for the set L_2. The Cartesian product of these two colorings will avoid long monochromatic arithmetic progressions with difference in L. To show the second statement, assume that for any finite coloring of the positive integers there are arbitrarily long arithmetic progressions with difference in L. Further assume that elements in different classes modulo n always have different colors (we can always refine the coloring to make this true), then we will have arbitrarily long arithmetic progressions with difference in $L \cap n\mathbb{N}$. □

Remark 2. From the previous theorem we can conclude that $n\mathbb{N}$ is a ladder for any $n \in \mathbb{N}$ and that nonzero classes modulo n are not ladders. Note as well that L is a ladder if and only if the complement of L is not a ladder. In particular, all cofinite sets are ladders and finite sets are not.

The best way to construct non trivial examples of ladders is using the polynomial van der Waerden theorem.

Theorem 3 (*Polynomial van der Waerden*). *Let $p_1, \ldots, p_k$ be polynomials with integer coefficients and no constant term. Then for any r-coloring of $\mathbb{N}$ there exist a, d such that $a, a + p_1(d), \ldots, a + p_k(d)$ are the same color.*

See [1,9], the latter for an elementary proof.

Corollary 4. *Let p be a polynomial with integer coefficients and no constant term. Then for any r-coloring of $\mathbb{N}$ and any $k \geq 1$ there exist a, d such that $a, a + p(d), \ldots, a + kp(d)$ are the same color.*

Proof. This follows from the polynomial van der Waerden by choosing $p_i = ip$, $1 \leq i \leq k$. □

In particular, if p is a polynomial with integer coefficients and satisfying the condition $p(0) = 0$, then the set $|p(\mathbb{N})| = p(\mathbb{N}) \cap \mathbb{N}$ is a ladder.

Question 2. *Is there a ladder which does not contain a set of the form $|p(\mathbb{N})|$ for some polynomial p as above?*

2. An elementary construction of sets which are not ladders

In this section we construct a family of sets that are not ladders. We shall discuss more properties of this family of sets in the following section.

Theorem 5. *Let $S = \left\{m^{2k-1} : k \in \mathbb{N}\right\}$. For $m \geq 5$ the set $S - S$ is not a 3-ladder.*

Proof. Let $f_i(n)$ be the ith digit in the base m representation of n. Consider the following coloring of the positive integers with 3 colors:

$$\alpha(n) = |\{i : f_{2i}(n) = 2\}| \mod 3.$$

Let $\{x_1, x_2, \ldots, x_n\}$ be an n-term monochromatic arithmetic progression with difference $d = m^{2j-1} - m^{2k-1}$. Suppose that $n > m^2 + m + 1$. Notice that

$$\left\{\left(f_{2k}(x_s), f_{2k-1}(x_s)\right) : 1 \leq s \leq m^2\right\} = \left\{0, \ldots, m-1\right\}^2.$$

Take $1 \leq t \leq m^2$ such that

$$\left(f_{2k}(x_t), f_{2k-1}(x_t)\right) = (3, 0).$$

We can calculate the values of $\left(f_{2k}(x_{t+i}), f_{2k-1}(x_{t+i})\right)$ for $0 \leq i \leq m+1$; in particular,

$$\begin{aligned} \left(f_{2k}(x_{t+1}), f_{2k-1}(x_{t+1})\right) &= (2, m-1), \\ \left(f_{2k}(x_{t+m}), f_{2k-1}(x_{t+m})\right) &= (2, 0), \\ \left(f_{2k}(x_{t+m+1}), f_{2k-1}(x_{t+m+1})\right) &= (1, m-1). \end{aligned}$$