European Journal of Combinatorics 50 (2015) 30–37

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Bijections between affine hyperplane arrangements and valued graphs

Sylvie Corteel [a], David Forge [b], Véronique Ventos [b]

[a] *LIAFA, CNRS et Université Paris Diderot 7, Case 7014, 75205 Paris cedex 13, France*

[b] *Laboratoire de recherche en informatique UMR 8623, Bât. 650, Université Paris-Sud, 91405 Orsay Cedex, France*

ARTICLE INFO

Article history:
Available online 28 May 2015

Dedicated to Michel Las Vergnas

ABSTRACT

We show new bijective proofs of previously known formulas for the number of regions of some deformations of the braid arrangement, by means of a bijection between the no-broken-circuit sets of the corresponding integral gain graphs and some kinds of labelled binary trees. This leads to new bijective proofs for the Shi, Catalan, and similar hyperplane arrangements.

1. Introduction

An *integral gain graph* is a graph whose edges are labelled invertibly by integers; that is, reversing the direction of an edge negates the label (the *gain* of the edge). The *affinographic hyperplane arrangement*, $\mathcal{A}[\Phi]$, that corresponds to an integral gain graph Φ is the set of all hyperplanes in $\mathbb{R}^n$ of the form $x_j - x_i = g$ for edges (i, j) with $i < j$ and gain g in Φ. (See [14, Section IV.4.1, pp. 270–271] or [7].)

In recent years there has been much interest in real hyperplane arrangements of this type, such as the braid arrangement, the Shi arrangement, the Linial arrangement, and the composed-partition or Catalan arrangement. For all these families, the characteristic polynomials and the number of regions have been found [12]. For the Shi, Braid, Linial and Catalan arrangements, it is known that the regions are in bijection with certain labelled trees or parking functions. See [1,2,10,9,11,13] and references therein.

In this paper we give bijective proofs of the number of regions for some of these arrangements by establishing bijections between the no-broken-circuit (NBC) sets and types of labelled trees and

E-mail addresses: corteel@liafa.univ-paris-diderot.fr (S. Corteel), forge@lri.fr (D. Forge), ventos@lri.fr (V. Ventos).

http://dx.doi.org/10.1016/j.ejc.2015.04.003
0195-6698/

forests, which can be counted directly. This means that we use the fact that the number of regions is equal to the number of NBC sets. This idea allows us to give a bijection between regions of hyperplane arrangements defined by $x_j - x_i = g$ with $g \in [a, b]$ and $a+b = 0$ or $a+b = 1$; that is the hyperplane arrangements of the type "extended braid" and "extended Shi".

The paper is organized as follows. In Section 2, we give some basic definitions. In Section 3, we define the core idea; that is the definition of the height function. In Section 4, we present NBC sets and trees. In Section 5, we characterize the NBC trees and use this characterization in Section 6 to give a bijection between the NBC-trees and certain labelled trees. In Section 7, we highlight two special cases and we end in Section 8 with some concluding remarks.

2. Basic definitions

An *integral gain graph* $\Phi = (\Gamma, \varphi)$ consists of a graph $\Gamma = (V, E)$ and an orientable function $\varphi : E \to \mathbb{Z}$, called the *gain mapping*. Orientability means that, if (i, j) denotes an edge oriented in one direction and (j, i) the same edge with the opposite orientation, then $\varphi(j, i) = -\varphi(i, j)$. We have no loops but multiple edges are permitted. For the rest of the paper, we denote the vertex set by $V = \{1, 2, \dots, n\} =: [n]$ with $n \geq 1$. We use the notations (i, j) for an edge with endpoints i and j, oriented from i to j, and $g(i, j)$ for such an edge with gain g; that is, $\varphi(g(i, j)) = g$. (Thus $g(i, j)$ is the same edge as $(-g)(j, i)$. The edge $g(i, j)$ corresponds to a hyperplane whose equation is $x_j - x_i = g$.) A *circle* is a connected 2-regular subgraph, or its edge set. Writing a circle C as a word $e_1 e_2 \cdots e_l$, the gain of C is $\varphi(C) := \varphi(e_1) + \varphi(e_2) + \cdots + \varphi(e_l)$; then it is well defined whether the gain is zero or nonzero. A subgraph is called *balanced* if every circle in it has gain zero. We will consider most especially balanced circles.

Given a linear order $<_O$ on the set of edges E, a *broken circuit* is the set of edges obtained by deleting the smallest element in a balanced circle. A set of edges, $N \subseteq E$, is a *no-broken-circuit set* (NBC set for short) if it contains no broken circuit. This notion from matroid theory (see [3] for reference) is very important here. We denote by $\mathcal{N}$ the set of NBC sets of the gain graph. It is well known that this set depends on the choice of the order, but its cardinality does not.

We can now transpose some ideas or problems from hyperplane arrangements to gain graphs. For any integers a, b, n, let K_n^{ab} be the gain graph built on vertices $V = [n]$ by putting on every edge (i, j) all the gains k, for $a \leq k \leq b$. These gain graphs are expansion of the complete graph and their corresponding arrangements are called sometimes deformations of the braid arrangement, truncated arrangements or affinographic arrangements. We have four main examples coming from well known hyperplane arrangements. We denote by B_n the gain graph K_n^{00} and call it the *braid gain graph*, by L_n the gain graph K_n^{11} and call it the *Linial gain graph*, by S_n the gain graph K_n^{01} and call it the *Shi gain graph* and finally by C_n the gain graph K_n^{-11} and call it the *Catalan gain graph*.

3. Height

We introduce the notion of height function on an integral gain graph on the vertex set $[n]$. A height function h defines two important things for the rest of the paper: the induced gain subgraph $\Phi[h]$ of a gain graph Φ and an order O_h on the set of vertices extended lexicographically to the set of edges.

Definition 1. A *height function* on a set V is a function h from V to $\mathbb{N}$ (the natural numbers including 0) such that $h^{-1}(0) \neq \emptyset$. The *corner* of the height function is the smallest element of greatest height.

Let Φ be a connected and balanced integral gain graph on a set V of integers. The *height function* of the gain graph is the unique height function h_Φ such that for every edge $g(i, j)$ we have $h_\Phi(j) - h_\Phi(i) = g$. (Such a function exists iff Φ is balanced.) The *corner of* Φ is the corner of h_Φ.

We say that an edge $g(i, j)$ is *coherent with* h if $h(j) - h(i) = g$.

Definition 2. Let Φ be a gain graph also on $V = [n]$ and h be a height function on V. The subgraph $\Phi[h]$ of Φ *selected by* h is the gain subgraph on the same vertex set V whose edges are the edges of Φ that are coherent with h.