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The transition matroid of a 4-regular graph: An introduction

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ABSTRACT

Given a 4-regular graph F, we introduce a binary matroid $M_\tau(F)$ on the set of transitions of F. Parametrized versions of the Tutte polynomial of $M_\tau(F)$ yield several well-known graph and knot polynomials, including the Martin polynomial, the homflypt polynomial, the Kauffman polynomial and the Bollobás–Riordan polynomial.

1. Introduction

A graph is determined by two finite sets, one set containing vertices and the other containing edges. Each edge is incident on one or two vertices; an edge incident on only one vertex is a loop. We think of an edge as consisting of two distinct half-edges, each of which is incident on precisely one vertex. In this paper we are especially interested in 4-regular graphs, i.e., graphs in which each vertex has precisely four incident half-edges. The special theory of 4-regular graphs was initiated by Kotzig and although his definitions and results have been generalized and modified over the years, most of the basic ideas of the theory appear in his seminal paper [50].

Matroids were introduced by Whitney [81], and there are several standard texts about them [36,61,77–80]. In this paper we will only encounter binary matroids. If M is a $GF(2)$-matrix with columns indexed by the elements of a set S, then the binary matroid represented by M is given by defining the rank of each subset $A \subseteq S$ to be equal to the dimension of the $GF(2)$-space spanned by the corresponding columns of M. Matroids can be defined in many other ways. In particular, the minimal nonempty subsets of S that correspond to linearly dependent sets of columns of M are the circuits of the matroid represented by M. We will not refer to matroid circuits often, to avoid confusion with the following definition.

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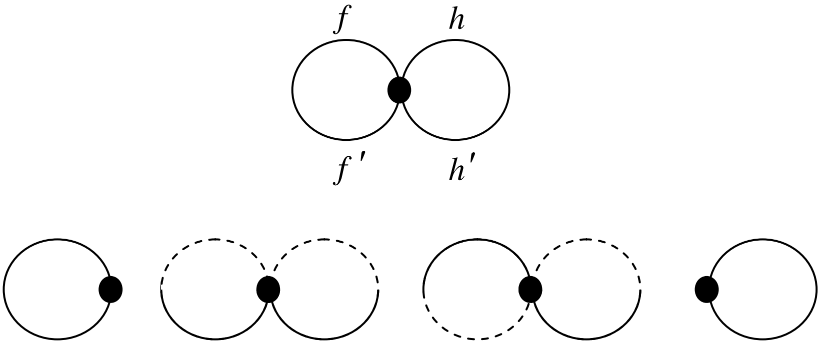

Fig. 1. A 4-regular graph with one vertex has four distinct circuits.

A circuit in a graph is a sequence $v_1, h_1, h_1', v_2, h_2, \ldots, h_k,\ h_k' = h_0', v_{k+1} = v_1$ such that for each $i \in \{1, \ldots, k\}$, h_i and h_i' are half-edges of a single edge and h_{i-1}' and h_i are both incident on v_i. The half-edges that appear in a circuit must be pairwise distinct, but vertices may be repeated. Two circuits are considered to be the same if they differ only by a combination of cyclic permutations $(1, \ldots, k) \mapsto (i, \ldots, k, 1, \ldots, i-1)$ and reversals $(1, \ldots, k) \mapsto (k, \ldots, 1)$. Notice that these definitions seem to be essentially non-matroidal: circuits may be nested, and distinct circuits may involve precisely the same vertices and half-edges, in different orders. For instance if a graph has one vertex v and two edges $e_1 = \{f, f'\}$ and $e_2 = \{h, h'\}$ (both loops), then it has four different circuits: v, f, f', v; v, f, f', v, h', h, v; v, f, f', v, h, h', v and v, h, h', v. See Fig. 1, where these circuits are indicated from left to right, using the convention that when a circuit traverses a vertex, the dash style (dashed or undashed) is maintained.

A circuit $v_1, h_1, h_1', v_2, \ldots, h_k, h_k' = h_0', v_{k+1} = v_1$ in a 4-regular graph is specified by the triples h_{i-1}', v_i, h_i where h_{i-1}' and h_i are distinct half-edges incident on v_i. We call such a triple a *single transition*. Kotzig called these triples "transitions" [50], but we adopt the convention used by other authors (including Ellis-Monaghan and Sarmiento [31], Jaeger [42] and Las Vergnas [52,53]) that a *transition* consists of two disjoint single transitions at the same vertex.

A circuit partition (or Eulerian partition or ξ-decomposition) of a 4-regular graph F is a partition of $E(F)$ into edge-disjoint circuits. These partitions were mentioned by Kotzig [50], and since then it has become clear that they are of fundamental significance in the theory of 4-regular graphs. Expanding on earlier work of Martin [57], Las Vergnas [53] introduced the generating function that records the sizes of circuit partitions of F, and also the generating functions that record the sizes of directed circuit partitions of directed versions of F; he called these generating functions the Martin polynomials of F. A circuit partition of F is determined by choosing one of the three transitions at each vertex, and Jaeger [42] used this fact in defining his transition polynomial, a form of the Martin polynomial that incorporates transition labels. A labeled form of the Martin polynomial was independently discovered by Kauffman, who used it in his bracket polynomial definition of the Jones polynomial of a knot or link [45,47].

For plane graphs, there is an indirect connection between Martin polynomials and graphic matroids, introduced by Martin [57] and further elucidated by Las Vergnas [51] and Jaeger [42]. (The corresponding result for the Kauffman bracket is due to Thistlethwaite [68].) The complementary regions of a 4-regular graph F embedded in the plane can be colored checkerboard fashion, yielding a pair of dual graphs with F as medial; the cycle matroid of either of the two dual graphs yields the Martin polynomial of a directed version of F. This indirect connection has been extended to several formulas, each time weakening the connection with matroids: Jaeger extended it to include information from the undirected Martin polynomial [40], Las Vergnas extended it to medial graphs in the projective plane and the torus [52], and Ellis-Monaghan and Moffatt extended it to include medials in surfaces of all genera [28,29].

The purpose of the present paper is to introduce a more general connection between matroids and Martin polynomials, which holds for all 4-regular graphs and does not require surface geometry.