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On inversion sets and the weak order in Coxeter groups

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ABSTRACT

In this article, we investigate the existence of joins in the weak order of an infinite Coxeter group W. We give a geometric characterization of the existence of a join for a subset X in W in terms of the inversion sets of its elements and their position relative to the imaginary cone. Finally, we discuss inversion sets of infinite reduced words and the notions of biconvex and biclosed sets of positive roots.

1. Introduction

The Cayley graph of a Coxeter system (W, S) is naturally oriented: we orient an edge $w \to ws$ if $w \in W$ and $s \in S$ such that $\ell(ws) > \ell(w)$. Here $\ell(u)$ denotes the *length* of a reduced word for $u \in W$ over the alphabet S. The Cayley graph of (W, S) with this orientation is the Hasse diagram of the *(right) weak order* $\leq$, see [4, Chapter 3] for definitions and background.

The weak order encodes a good deal of the combinatorics of reduced words associated to W. For instance for $u, v \in W$, $u \leq v$ if and only if a reduced word for u is a prefix for a reduced word for v. Moreover, Björner [3, Theorem 8] shows that the poset $(W, \leq)$ is a complete meet semilattice: for any $A \subseteq W$, there exists an infimum $\bigwedge A \in W$, also called the *meet* of A, see also [4, Theorem 3.2.1]. This means, in particular, that any $u, v \in W$ have a common greatest prefix, that is, the unique longest $g = u \wedge v \in W$ for which a reduced word is the prefix of a reduced word for u and of a reduced word for v.

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In the case of finite Coxeter systems, the weak order turns out to be a complete ortholattice [3, Theorem 8], thanks to the existence of the unique longest element $w_\circ$ in W: $u \leq v$ if and only if $uw_\circ \geq vw_\circ$ and the supremum of $A \subseteq W$ exists and is $\bigvee A = (\bigwedge(Aw_\circ))w_\circ$, also called the *join of A*. In particular, for any $u, v \in W$, there exists a unique smallest $g = u \vee v \in W$ with two reduced words, one with prefix a reduced word for u and the other with prefix a reduced word for v. The existence of $w_\circ$ and the fact that the weak order is a lattice play important roles in the study of structures related to (W, S) such as Cambrian lattices and cluster combinatorics [31–33], Garside elements in spherical Artin-braid groups, see for instance [11], or reflection orders, their initial sections and Kazhdan–Lusztig polynomials, see the discussion in [16, §2].

When trying to generalize this technology to infinite Coxeter systems, the absence of $w_\circ$ and of a join in general is crucially missed. For instance the Coxeter sortable elements and Cambrian fans fail to recover the whole cluster combinatorics [34, §1.2]. Examples suggest that we need to adjoin new elements to the weak order of (W, S) in order to generalize properly its combinatorial usage to infinite Coxeter groups. In other words, we need to define a larger family of objects, containing the elements of the infinite Coxeter groups, that would further have greatest common prefixes (meet) and least common multiples (join). This brings us to the following question: is there, for each infinite Coxeter system, a suitable complete ortholattice with the usual weak order as a subposet?

As a first natural candidate to consider, one could think of the set of finite and infinite reduced words over S, modulo finite and infinite sequences of relations, but this fails already in the case of infinite dihedral groups since the two infinite words do not have a join in that case. Further, a completion was obtained by Dyer [15, Corollary 10.8] using the theory of rootoids, but it is not suitable to our combinatorial need. Hereafter, we investigate an extension of the weak order, containing the set of infinite reduced words, proposed by M. Dyer and conjectured to be a complete ortholattice [16, Conjecture 2.5].

This extension uses the notion of *biclosed subsets* of positive roots, a suitable generalizations of inversion sets of words, which we recall in Section 2. This conjecture is still open. The main difficulties are the following:

(1) understand biclosed sets in general. Finite biclosed sets are the inversion sets of the elements of W, see Section 2, but what about the other biclosed sets?
(2) understand the possible candidates for a join.

In this article we investigate these two points. After surveying what is known about the notions of biclosed sets and inversion sets in Section 2, we give in Section 3 a geometric criterion for the existence of a join in $(W, \leq)$: the join of a subset $X \subseteq W$ exists if and only if X is finite and the cone spanned by the inversion sets of the elements in X is strictly separated from the *imaginary cone* – the conic hull of the limit roots –, see Theorem 3.2. As a corollary, we obtain that a subset A of positive roots is a finite biclosed set if and only if A is the set of roots contained in a closed halfspace that does not intersect the imaginary cone (Corollary 3.4). The proof is based on a characterization of the existence of the join obtained by Dyer [16] and on the study of limit roots and imaginary cones started in [21,17,19]. In Section 4, we use our geometric criterion on finite biclosed sets to give a (partially conjectural) characterization of biclosed sets corresponding to the inversion sets of infinite words, see Corollary 4.4 and Conjecture 2. This characterization extends to arbitrary Coxeter systems a result of P. Cellini and P. Papi [8] valid for affine Coxeter systems (see Remark 4.6). Finally, in Section 5 we discuss the similarities and especially the *differences* between the notions of *biclosed, biconvex* and *separable* subsets of positive roots. The notion of biconvex sets in particular was used in recent works [2,28] related to the affine cases in which the authors also conjectured an extension of the weak order similar to Dyer's conjecture.

2. Biclosed sets, inversion sets and join in the weak order

2.1. Geometric representation of a Coxeter system

Let (V, B) be a quadratic space: V is a finite-dimensional real vector space endowed with a symmetric bilinear form B. The group of linear maps that preserves B is denoted by $O_B(V)$. The *isotropic*