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Maximum degree in minor-closed classes of graphs

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ABSTRACT

Given a class of graphs $\mathcal{G}$ closed under taking minors, we study the maximum degree Δ_n of random graphs from $\mathcal{G}$ with n vertices. We prove several lower and upper bounds that hold with high probability. Among other results, we find classes of graphs providing orders of magnitude for Δ_n not observed before, such us $\log n/\log\log\log n$ and $\log n/\log\log\log\log n$.

1. Introduction

A class of labeled graphs $\mathcal{G}$ is minor-closed if whenever a graph G is in $\mathcal{G}$ and H is a minor of G, then H is also in $\mathcal{G}$. A basic example is the class of planar graphs or, more generally, the class of graphs embeddable in a fixed surface.

All graphs in this paper are labeled. Let $\mathcal{G}_n$ be the graphs in $\mathcal{G}$ with n vertices. By a random graph from $\mathcal{G}$ of size n we mean a graph drawn with uniform probability from $\mathcal{G}_n$. We say that an event A in the class $\mathcal{G}$ holds with high probability (w.h.p.) if the probability that A holds in $\mathcal{G}_n$ tends to 1 as $n \to \infty$. Let Δ_n be the random variable equal to the maximum vertex degree in random graphs from $\mathcal{G}_n$. We are interested in events of the form

$$\Delta_n \le f(n) \quad \text{w.h.p.}$$

and of the form

$$\Delta_n \ge f(n) \quad \text{w.h.p.}$$

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Typically $f(n)$ will be of the form $c \log n$ for some constant c, or some related functions. We say that $f(n) = O(g(n))$ if there exist an integer n_0 and a constant $c > 0$ such that $|f(n)| \le c|g(n)|$ for all $n \ge n_0$, $f(n) = \Omega(g(n))$, if $g(n) = O(f(n))$, and finally $f(n) = \Theta(g(n))$, if both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ hold. Also, $f(n) = \omega(g(n))$, if $\lim_{n\to\infty} |f(n)|/|g(n)| = \infty$, and $f(n) = o(g(n))$, if $g(n) = \omega(f(n))$. Throughout this paper $\log n$ refers to the natural logarithm.

A classical result says that for labeled trees Δ_n is of order $\log n/\log\log n$ (see [13]). In fact, much more precise results are known in this case, in particular that (see [2])

$$\frac{\Delta_n}{\log n/\log\log n} \to 1 \quad \text{in probability.}$$

Many more results about the distribution of maximum degree, its concentration, and several different models of randomly generated trees can be found in the survey of [9].

McDiarmid and Reed [12] show that for the class of planar graphs there exist constants $0 < c_1 < c_2$ such that

$$c_1 \log n < \Delta_n < c_2 \log n \quad \text{w.h.p.}$$

More recently this result has been strengthened using subtle analytic and probabilistic methods [5], by showing the existence of a computable constant c such that

$$\frac{\Delta_n}{\log n} \to c \quad \text{in probability.}$$

For planar maps (planar graphs with a given embedding), more precise results on the distribution of Δ_n can be found in [7,3,8].

Analogous results have been proved for series–parallel and outerplanar graphs [4], with suitable constants. Using the framework of Boltzmann samplers, results about the degree distribution of subcritical graph classes such as outerplanar graphs, series–parallel graphs, cactus graphs and clique graphs can also be found in [1]. This paper also contains conjectures of the exact values of c_{OP} (c_{SP}, respectively) so that the maximum degree in outerplanar graphs (series–parallel graphs, respectively) will be roughly $c_{OP} \log n$ ($c_{SP} \log n$, respectively).

The goal in this paper is to analyze the maximum degree in additional minor-closed classes of graphs. Our main inspiration comes from the work of McDiarmid and Reed mentioned above. The authors develop proof techniques based on double counting that assume only mild conditions on the classes of graphs involved. We now explain the basic principle.

Let $\mathcal{G}$ be a class of graphs and suppose we want to show that a property P holds in $\mathcal{G}$ w.h.p. Let $\mathcal{B}_n$ be the graphs in $\mathcal{G}_n$ that do not satisfy P (the 'bad' graphs). Suppose that for a constant fraction $\alpha > 0$ of graphs in $\mathcal{B}_n$ we have a rule producing at least $C(n)$ graphs in $\mathcal{G}_n$ (the 'construction' function). A graph in $\mathcal{G}_n$ can be produced more than once, but assume every graph in $\mathcal{G}_n$ is produced at most $R(n)$ times (the 'repetition' function). By double counting we have

$$\alpha|\mathcal{B}_n|C(n) \le |\mathcal{G}_n|R(n),$$

hence

$$\alpha\frac{|\mathcal{B}_n|}{|\mathcal{G}_n|} \le \frac{R(n)}{C(n)}.$$

If the procedure is such that $C(n)$ grows faster than $R(n)$, that is $R(n) = o(C(n))$, then we conclude that $|\mathcal{B}_n| = o(|\mathcal{G}_n|)$, that is, the proportion of bad graphs goes to 0. Equivalently, property P holds w.h.p. We often use the equivalent formulation $C(n)/R(n) \to \infty$.

We will apply this principle in order to obtain lower and upper bounds on the maximum degree for several classes. In this context, lower bounds are easier to obtain, and only in some cases we are able to prove matching upper bounds. The proof of the upper bound for planar graphs in [12] depends very strongly on planarity, and it seems difficult to adapt it to general situations; however we obtain such a proof for outerplanar graphs. On the other hand, we develop new tools for proving upper bounds based on the decomposition of a connected graph into 2-connected components.