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There are no finite partial cubes of girth more than 6 and minimum degree at least 3

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ABSTRACT

Partial cubes are graphs isometrically embeddable into hypercubes. We analyze how isometric cycles in partial cubes behave and derive that every partial cube of girth more than 6 must have vertices of degree less than 3. As a direct corollary we get that every regular partial cube of girth more than 6 is an even cycle. Along the way we prove that every partial cube G with girth more than 6 is a tree-zone graph and therefore $2n(G) - m(G) - i(G) + ce(G) = 2$ holds, where $i(G)$ is the isometric dimension of G and $ce(G)$ its convex excess.

1. Introduction

Graphs that can be isometrically embedded into hypercubes are called partial cubes. They form a well known class of graphs which inherits many structural properties from hypercubes. For this reason, they were introduced by Graham and Pollack [16] as a model for interconnection networks and later found different applications, for examples see [2,13,22]. There has been much theory developed about partial cubes, we direct an interested reader to books [11,17] and the survey [26]. For recent results in the field, see [1,8,9,15,29].

Probably the best known subfamily of partial cubes are median graphs [3,17,21]. Many questions that are currently open for partial cubes are long answered for median graphs. Comparing to median graphs, we can learn a lot about partial cubes and even predict certain properties. An example of this is the topic of classifying regular graphs in each class. It was Mulder [25] who already in 1980 showed that hypercubes are the only finite regular median graphs; this result has been in some instances generalized also to infinite graphs [4,18,23,24]. On the other hand, it seems very difficult to find

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(non-median) regular partial cubes (particularly in the cubic case), extensive studies have been made in [6,7,12,19]. In fact, all known cubic partial cubes are planar, besides the Desargues graph [19]. One of the motivations for this article is to find out why this is so.

One of the most important differences between partial cubes and median graphs is hidden in the cycles of these graphs, particularly in the behavior of isometric and convex cycles. The convex closure of an isomeric cycle in a median graph is a hypercube (for a proof and a generalization to a larger subclass of partial cubes, see [27]). This implies that median graphs that are not trees have girth four, which is far from true in partial cubes. It is an interesting fact that all the known examples of regular partial cubes have girth four, with the exception of even cycles and the middle level graphs (which have girth 6). This motivates the analysis of partial cubes of higher girths.

A motivation for the study of partial cubes with high minimum degree comes from the theory of oriented matroids. Every oriented matroid is characterized by its tope graph, formed by its maximal covectors [5]. It is a well known fact that tope graphs are partial cubes, while there is no good characterization of partial cubes that are tope graphs [14]. It follows from basic properties of oriented matroids that the minimum degree of a tope graph is at least the rank of the oriented matroid it describes. Since the tope graphs of oriented matroids with rank at most 3 are characterized [14], there is a special interest in graphs with high minimum degree.

Klavžar and Shpectorov [20] proved a certain ''Euler-type'' formula for partial cubes, concerning convex cycles. Moreover, they defined the zone graphs of a partial cube: graphs that emerge if we consider how convex cycles in a partial cube intersect. The latter gave motivation to analyze the space of isometric cycles in partial cubes.

The main contribution of this paper is a theorem which shows that there are no finite partial cubes of girth more than 6 and minimum degree at least 3. This helps to understand why it is difficult to find regular partial cubes, since it implies that, besides even cycles, there are none with girth more than 6. To prove the theorem we introduce two concepts – a traverse of isometric cycles and intertwining of isometric cycles – and show some properties of them. We hope that these two definitions will give a new perspective on partial cubes.

In the rest of this section basic definitions and results needed are given. We will consider only simple (possibly infinite) graphs in this paper. The *Cartesian product* $G\,\square\,H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$ and the edge set consisting of all pairs $\{(g_1, h_1), (g_2, h_2)\}$ of vertices with $\{g_1, g_2\} \in E(G)$ and $h_1 = h_2$, or $g_1 = g_2$ and $\{h_1, h_2\} \in E(H)$. *Hypercubes* or *n-cubes* are Cartesian products of n-copies of K_2. We say a subgraph H of G is *convex* if for every pair of vertices in H also every shortest path connecting them is in H. On the other hand, a subgraph is *isometric* if for every pair of vertices in H also some shortest path connecting them is in H. A *partial cube* is a graph that is isomorphic to an isometric subgraph of some hypercube.

For a graph G, we define the relation Θ on the edges of G as follows: $ab\Theta xy$ if $d(a, x) + d(b, y) \neq d(a, y) + d(b, x)$, where d is the shortest path distance function. In partial cubes Θ is an equivalence relation (in fact a bipartite graph is a partial cube if and only if Θ is an equivalence relation [30]), and we write F_{uv} for the set of all edges that are in relation Θ with uv. We define W_{uv} as the subgraph induced by all vertices that are closer to vertex u than to v, that is $W_{uv} = \langle\{w : d(u, w) < d(v, w)\}\rangle$. In a partial cube G, subgraphs W_{uv} are convex, and the sets $V(W_{uv})$ and $V(W_{vu})$ partition $V(G)$, with F_{uv} being the set of edges joining them. We define U_{uv} to be the subgraph induced by the set of vertices in W_{uv} which have a neighbor in W_{vu}. For details and further results, see [17].

We shall need a few simple results about partial cubes. If $u_1v_1\Theta u_2v_2$ with $u_2 \in U_{u_1v_1}$, then $d(u_1, u_2) = d(v_1, v_2) = d(u_1, v_2) - 1 = d(u_2, v_1) - 1$. A path P is a shortest path or a *geodesic* if and only if it has all of its edges in pairwise different Θ classes. For fixed u, v all shortest u, v-paths pass the same Θ-classes of G. If C is a cycle and e an edge on C, then there is another edge on C in relation Θ with e. We denote with $I(a, b)$ the *interval* from vertex a to vertex b, i.e. the induced subgraph on all the vertices that lie on some shortest a, b-path. In a partial cube, for every vertices a and b, the subgraph $I(a, b)$ is convex. For the details, we again refer to [17].

In [20], the following definition was given: Let G be a partial cube and F be some equivalence class of relation Θ. The *F-zone graph*, denoted with Z_F, is the graph with $V(Z_F) = F$, vertices f and f' being adjacent in Z_F if they belong to a common convex cycle of G. We call a partial cube whose all zone graphs are trees a *tree-zone* partial cube.