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A sufficient local degree condition for Hamiltonicity in locally finite claw-free graphs

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ABSTRACT

Among the well-known sufficient degree conditions for the Hamiltonicity of a finite graph, the condition of Asratian and Khachatrian is the weakest and thus gives the strongest result. Diestel conjectured that it should extend to locally finite infinite graphs G, in that the same condition implies that the Freudenthal compactification of G contains a circle through all its vertices and ends. We prove Diestel's conjecture for claw-free graphs.

1. Introduction

Problems concerning the existence of Hamilton cycles in finite graphs are studied quite a lot, but to decide whether a given finite graph is Hamiltonian is difficult. Nevertheless, or even because of that, many sufficient or necessary conditions for Hamiltonicity have been found which are often easy to handle. One common class of sufficient conditions are degree conditions. An early result in this area is the following theorem of Dirac [11].

Theorem 1.1 ([11, Thm. 3]). *Every finite graph with $n \geq 3$ vertices and minimum degree at least $n/2$ is Hamiltonian.*

The next result generalizes the theorem of Dirac and is due to Ore [16].

Theorem 1.2 ([16, Thm. 2]). *Let G be a finite graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n$ for any two non-adjacent vertices u and v of G, then G is Hamiltonian.*

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Both of these theorems state sufficient conditions for Hamiltonicity which involve the total number of vertices in the given graph. So we could say that both conditions do not have a local form. Furthermore, these conditions imply that the considered graphs have diameter at most 2. In contrast to this, the next theorem, which is due to Asratian and Khachatrian [1], generalizes both theorems above and allows graphs of arbitrary diameter. In order to state the theorem, we need the following local property of a graph:

$$d(u) + d(w) \geq |N(u) \cup N(v) \cup N(w)| \quad \textit{for every induced path } uvw. \qquad (*)$$

The theorem of Asratian and Khachatrian can now be formulated as follows:

Theorem 1.3 (*[1, Thm. 1]*)**.** *Every finite connected graph which satisfies (∗) and has at least three vertices is Hamiltonian.*

The vast majority of Hamiltonicity results deal only with finite graphs, since it is not clear what a Hamilton cycle in an infinite graph should be. We follow the topological approach initiated by Diestel and Kühn [8–10] and further outlined in [5,6], which solves this problem in a reasonable way by using as infinite cycles of a graph G the circles in its Freudenthal compactification $|G|$. Then circles which use infinitely many vertices of G are possibly to exist. Now the notion of a Hamiltonicity extends in an obvious way: Call a locally finite connected graph G *Hamiltonian* if there is a circle in $|G|$ that contains all vertices of G.

Some Hamiltonicity results for finite graphs have already been generalized to locally finite graphs using this notion but not all of them are complete generalizations. Theorems that involve local conditions as in Theorem 1.3 are more likely to generalize to locally finite graphs since they are still well-defined for infinite graphs and might allow compactness arguments. For results in this field, see [2,4,12–15].

This paper deals with a conjecture of Diestel [6, Conj. 4.13] about Hamiltonicity which says that Theorem 1.3 can be generalized to locally finite graphs. The main result of this paper is the following theorem, which shows that the conjecture of Diestel holds for claw-free graphs where we call a graph *claw-free* if it does not contain the claw, i.e., the graph $K_{1,3}$, as an induced subgraph.

Theorem 1.4. *Every locally finite, connected, claw-free graph which satisfies (∗) and has at least three vertices is Hamiltonian.*

The rest of this paper is structured in the following way. In Section 2 we recall some basic definitions and introduce some notation we shall need in this paper. Section 3 contains some facts and lemmas which are needed in the proof of our main result. In the last section, Section 4, we consider locally finite graphs which satisfy condition (∗). There we give two infinite classes of examples of locally finite graphs satisfying (∗). In one class, all members are claw-free, while all elements of the other class have claws as induced subgraphs. The rest of Section 4 deals with the proof of Theorem 1.4. At the very end of the paper we discuss where we need the assumption of being claw-free for the proof of our main theorem.

2. Basic definitions and notation

In general, we follow the graph-theoretic notation of [5] in this paper. For basic graph-theoretic facts, we refer the reader also to [5]. Beside finite graph theory, a topologically approach to infinite locally finite graphs is covered in [5, Ch. 8.5]. For a survey in this field, we refer to [6].

All graphs considered in this paper are undirected and simple. Furthermore, a graph is not assumed to be finite. Now we fix an arbitrary graph $G = (V, E)$ for this section.

The graph G is called *locally finite* if every vertex of G has only finitely many neighbours.

For a vertex set X of G, we denote by $G[X]$ the induced subgraph graph of G whose vertex set is X. We write $G - X$ for the graph $G[V \setminus X]$, but for singleton sets, we omit the set brackets and write just $G - v$ instead of $G - \{v\}$ where $v \in V$. We denote the cut which consists of all edges of G that have one endvertex in X and the other endvertex in $V \setminus X$ by $\delta(X)$.