European Journal of Combinatorics 55 (2016) 119–143

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

A new discrete dynamical system of signed integer partitions

G. Cattaneo [a], G. Chiaselotti [b], P.A. Oliverio [b], F. Stumbo [c]

[a] *Department of Informatics, Systems and Communications, University of Milano-Bicocca, 20126 Milano, Italy*

[b] *Dipartimento di Matematica, Universitá della Calabria, Via Pietro Bucci, Cubo 30B, 87036 Arcavacata di Rende (CS), Italy*

[c] *Dipartimento di Matematica e Informatica, Universitá di Ferrara, Via Machiavelli 35, 44121, Ferrara, Italy*

ARTICLE INFO

Article history:
Received 8 May 2015
Accepted 12 February 2016
Available online 8 March 2016

ABSTRACT

In Brylawski (1973) Brylawski described the covering property for the domination order on non-negative integer partitions by means of two rules. Recently, in Bisi et al. (in press), Cattaneo et al. (2014), Cattaneo et al. (2015) the two classical Brylawski covering rules have been generalized in order to obtain a new lattice structure in the more general signed integer partition context. Moreover, in Cattaneo et al. (2014), Cattaneo et al. (2015), the covering rules of the above signed partition lattice have been interpreted as evolution rules of a discrete dynamical model of a two-dimensional p–n semiconductor junction in which each positive number represents a distribution of holes (positive charges) located in a suitable strip at the left semiconductor of the junction and each negative number a distribution of electrons (negative charges) in a corresponding strip at the right semiconductor of the junction. In this paper we introduce and study a new sub-model of the above dynamical model, which is constructed by using a single vertical evolution rule. This evolution rule describes the natural annihilation of a hole–electron pair at the boundary region of the two semiconductors. We prove several mathematical properties of such new discrete dynamical model and we provide a discussion of its physical properties.

E-mail addresses: cattang@live.it, cattang@fislab.disco.unimib.it (G. Cattaneo), chiaselotti@unical.it (G. Chiaselotti), oliverio@unical.it (P.A. Oliverio), f.stumbo@unife.it (F. Stumbo).

http://dx.doi.org/10.1016/j.ejc.2016.02.003

1. Introduction

1.1. The BGK and BGKV models

The set of all the (non-negative) partitions of a (positive) integer m can be equipped of a partial order, usually called *dominance order*. Let us recall it: if $w = (w_0, \ldots, w_{l-1})$ and $w' = (w'_0, \ldots, w'_{l-1})$ are two nonincreasing sequences of non-negative integers having sum $m \in \mathbb{N}$ and equal length l, then the dominance order is defined by the following binary relation:

$$w \geq w' \Leftrightarrow \sum_{j=0}^{i} w_j \geq \sum_{j=0}^{i} w'_j \quad \text{for every } i. \tag{1}$$

In [16] Brylawski proved that the corresponding poset is indeed a lattice, denoted by $L_B(m)$, bounded by the least element (1^m) and the greatest element (m). The relevant fact to our aims is that the condition of covering in the lattice $L_B(m)$ can be expressed in an equivalent form as a result of Proposition 2.3 of [16]. Moreover, an original and clearer formulation of the above condition of covering has been formulated by Green and Kleitman in [31] by using two particular rules which operate under suitable conditions: the vertical displacement of one unit, called V-*rule*, and the horizontal displacement of one unit, called H-*rule*. In what follows we denote briefly by BGK the names Brylawski, Green and Kleitman and we also use the term BGK *rules* to refer to both the H-rule and the V-rule.

In this paper we adopt the useful point of view of a *discrete dynamical system*, briefly DDS, (see [32]) for the lattice $L_B(m)$. A dynamical system is any mathematical model where some rules, usually called *evolution rules*, describe how a point in a geometrical space depends on time. To standardize our terminology to DDS language, we call *configuration* any integer partition of $L_B(m)$ and *evolution rules* the BGK rules.

There are two ways in which we can apply the evolution rules on a given configuration of $L_B(m)$: a *sequential* way (in this case we also refer to *sequential dynamic* of the DDS) and a *parallel* way (in this case we also refer to *parallel dynamic* of the DDS). In the sequential way, we apply both the V-rule and the H-rule separately on each summand of the configuration. In the parallel way, we apply the V-rule concurrently on all the summands of the configuration. When we start from the initial configuration and we apply the evolution rules in a sequential way, generally we obtain an oriented graph which represents the Hasse diagram of a poset of integer partitions with an order relation $\sqsubseteq$. In this case one usually proves that the relation w generates w', with some evolution rule, is equivalent to say that w covers w' with respect to the partial order $\sqsubseteq$. Therefore several concepts of the classical order theory find their interesting formulation in dynamic terms (for further details see also [12,14,15,17,19,23,24]).

In the case of $L_B(m)$, we choose an initial configuration (that is usually the maximum (m) of the lattice $L_B(m)$) and next we apply the BGK rules starting from this initial configuration and terminating in the configuration (1^m), which remains fixed under the BGK rule action. In this way the set of all configurations of $L_B(m)$ becomes a DDS, which we denote by $BGK(m)$. In $BGK(m)$, each summand w_j of an integer partition w is interpreted as a single column containing w_j movable blocks stacked at the site j of a one-dimensional array of columns (the Ferrer diagram of the partition). Then, the two Brylawski conditions characterizing the lattice covering of two integer partitions w, w' under dominance order are equivalently formalized as follows: a single block can slip from a pile to the next available one following particular horizontal and vertical shifts described by BGK rules. Such shifts generate a suitable variation of the integer partition, dynamically denoted by $w \rightarrow w'$, under the constraint of the invariance of their sum. In we admit only the V-rule action in the lattice $L_B(m)$, then we obtain a sub model $BGKV(m)$ of $BGK(m)$ whose mathematical and physical properties have been studied in [19].

1.2. The BGK model for signed partitions

A natural extension of the BGK system can be obtained by admitting also negative summands. In other words, we can introduce a new type of integer partitions having both positive summands and