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Spectral radius and fractional matchings in graphs



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ABSTRACT

A fractional matching of a graph G is a function f giving each edge a number in $[0, 1]$ so that $\sum_{e \in \Gamma(v)} f(e) \leq 1$ for each $v \in V(G)$, where $\Gamma(v)$ is the set of edges incident to v . The fractional matching number of G , written $\alpha'_*(G)$, is the maximum of $\sum_{e \in E(G)} f(e)$ over all fractional matchings f . Let G be an n -vertex connected graph with minimum degree d , let $\lambda_1(G)$ be the largest eigenvalue of G , and let k be a positive integer less than n . In this paper, we prove that if $\lambda_1(G) < d\sqrt{1 + \frac{2k}{n-k}}$, then $\alpha'_*(G) > \frac{n-k}{2}$. As a result, we prove $\alpha'_*(G) \geq \frac{nd^2}{\lambda_1(G)^2 + d^2}$; we characterize when equality holds in the bound.

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1. Introduction

Throughout this paper, all graphs are simple with no multiple edges and no loops. A matching of a graph G is a set of disjoint edges. The matching number of G , written $\alpha'(G)$, is the maximum size of a matching in G . The adjacency matrix $A(G)$ of G is the n -by- n matrix in which entry $a_{i,j}$ is 1 or 0 according to whether v_i and v_j are adjacent or not, where $V(G) = \{v_1, \dots, v_n\}$. The eigenvalues of G are the eigenvalues of its adjacency matrix $A(G)$. Let $\lambda_1(G), \dots, \lambda_n(G)$ be its eigenvalues in nonincreasing order. We call $\lambda_1(G)$ the spectral radius of G . In 2005, Brouwer and Haemers gave a sufficient condition on $\lambda_3(G)$ in an n -vertex connected k -regular graph G for the existence of a perfect matching in G . In fact, they proved that if G is k -regular and has no perfect matching, then G has at least three proper induced subgraphs H_1, H_2 , and H_3 , which are contained in the family \mathcal{H} of graphs, where $\mathcal{H} = \{H : |V(H)| \text{ is odd, and } k|V(H)| - k + 2 \leq 2|E(H)| \leq k|V(H)| - 1\}$. Note that if $H \in \mathcal{H}$,

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then H is not k -regular but may be an induced subgraph of a k -regular graph. Also, if k is even, then the upper bound on $2|E(H)|$ can be replaced by $k|V(H)| - 2$, since even regular graphs cannot have a cut-edge, where an edge e in a connected graph G is a *cut-edge* of G if $G - e$ is disconnected. By using the Interlacing Theorem [2,5], Lemma 1.6 [4] and the fact that the spectral radius of a graph is at least its average degree, they proved that if G is k -regular and has no perfect matching, then $\lambda_3(G) \geq \min_{i \in \{1,2,3\}} \lambda_1(H_i) > \min_{H \in \mathcal{H}} 2|E(H)|/|V(H)|$, where $H_i \in \mathcal{H}$. The bound on λ_3 that Brouwer and Haemers found could be improved, since equality in the bound on the spectral radius in terms of the average degree holds only when graphs are regular. Later, Cioabă, Gregory, and Haemers [3] found the minimum of $\lambda_1(H)$ over all graphs $H \in \mathcal{H}$. More generally, Cioabă and the author [4] determined connections between the eigenvalues of an l -edge-connected k -regular graph and its matching number when $1 \leq l \leq k - 2$.

As we mentioned above, the conditions on eigenvalues in the papers actually come from having determined the largest eigenvalue of a proper subgraph in the whole graph. Thus, the results may be improved. In this paper, we determine connections between the spectral radius of an n -vertex connected graph with minimum degree d and its fractional matching number. This connection comes from the whole graph and is best possible. A *fractional matching* of a graph G is a function f giving each edge a number in $[0, 1]$ so that $\sum_{e \in \Gamma(v)} f(e) \leq 1$ for each $v \in V(G)$, where $\Gamma(v)$ is the set of edges incident to v . The *fractional matching number* of G , written $\alpha'_*(G)$, is the maximum of $\sum_{e \in E(G)} f(e)$ over all fractional matchings f . In Section 3, we prove that if G is an n -vertex graph with minimum degree d and $\alpha'_*(G) \leq \frac{n-k}{2}$, then $\lambda_1(G) \geq d\sqrt{1 + \frac{2k}{n-k}}$. This result is best possible in the sense that there are graphs H with minimum degree d such that $\alpha'_*(H) = \frac{n-k}{2}$ and $\lambda_1(H) = d\sqrt{1 + \frac{2k}{n-k}}$, which is shown in Section 2.

2. Construction

In this section, we construct n -vertex connected graphs H with minimum degree d such that $\alpha'_*(H) = \frac{n-k}{2}$, and $\lambda_1(H) = d\sqrt{1 + \frac{2k}{n-k}}$.

Construction 2.1. Let d and k be positive integers. Let $\mathcal{H}(d, k)$ be the family of connected bipartite graphs H with the bipartitions A and B such that:

- (i) every vertex in A has degree d ,
- (ii) $|A| = |B| + k$, and
- (iii) the degrees of vertices in B are equal.

The condition (iii) of [Construction 2.1](#) guarantees that the degree of vertices in B is greater than d because $|A| > |B|$. Note that $\mathcal{H}(d, k)$ contains the complete bipartite graph $K_{d,d+k}$. In fact, there are more graphs in the family $\mathcal{H}(d, k)$.

Example 2.2. Let k be a positive integer. For $1 \leq i \leq k$, let H_i be a copy of $K_{d,d+1}$ with partite sets X_i and Y_i of sizes d and $d + 1$. Delete one edge in H_i . Restore the original vertex degrees by adding one edge joining Y_i and X_{i+1} for each i , with subscript taken modulo k . The resulting graph is G_k .

Let $A = \bigcup Y_i$ and $B = \bigcup X_i$. By the construction of G_k , it is connected and bipartite with the partitions A and B such that (i) every vertex in A has degree d , (ii) $|A| = |B| + k$, and (iii) every vertex in B has degree $d + 1$. Thus G_k is included in $\mathcal{H}(d, k)$. When k is even, we can also construct a graph included in $\mathcal{H}(d, k)$ from two copies of $K_{d,d+\frac{k}{2}}$, so there are many graphs in $\mathcal{H}(d, k)$, depending on k .

To compute the fractional matching number for graphs in $\mathcal{H}(d, k)$, we use a fractional analogue of the famous Berge–Tutte Formula [1] for the matching number. The *deficiency* of a vertex set S in a graph G , written $\text{def}_G(S)$ or simply $\text{def}(S)$, is $o(G - S) - |S|$, where $o(K)$ is the number of components of odd order of a graph K . The Berge–Tutte Formula is the equality $\alpha'(G) = \min_{S \subseteq V(G)} \frac{1}{2}(n - \text{def}(S))$, where $n = |V(G)|$. The special case $\alpha'(G) = n/2$ reduces to Tutte’s 1-Factor Theorem [8]: a graph G

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