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Exponential formulas for models of complex reflection groups

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ABSTRACT

In this paper we find exponential formulas for the Betti numbers of the De Concini–Procesi minimal wonderful models $Y_{G(r,p,n)}$ associated to the complex reflection groups $G(r, p, n)$. Our formulas are different from the ones already known in the literature: they are obtained by a new combinatorial encoding of the elements of a basis of the cohomology by means of set partitions with weights and exponents.

We also point out that a similar combinatorial encoding can be used to describe the faces of the real spherical wonderful models of type $A_{n-1}(=G(1, 1, n))$, B_n $(=G(2, 1, n))$ and $D_n(=G(2, 2, n))$. This provides exponential formulas for the f-vectors of the associated nestohedra: the Stasheff's associahedra (in this case closed formulas are well known) and the graph associahedra of type D_n.

1. Introduction

Let us start by fixing some notations. First we recall that the finite irreducible complex reflection groups, according to the Shephard–Todd classification (see [37]), are the groups $G(r, p, n)$, with $r, p, n \in \mathbb{Z}^+$ and $p|r$, plus 34 exceptional groups.

Let $C(r)$ be the cyclic group of order r generated by a primitive rth root of unity ζ. The group $G(r, 1, n)$, the full monomial group, is the wreath product of $C(r)$ and the symmetric group S_n. It can also be described as the group generated by all the complex reflections in $GL(\mathbb{C}^n)$ whose reflecting hyperplanes are the hyperplanes with equations $x_i = \zeta^\alpha x_j$, where $\alpha = 0, \ldots, r-1$, and $x_i = 0$.

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Its elements are all the linear transformations $g(\sigma, \epsilon) : \mathbb{C}^n \to \mathbb{C}^n$ defined on the standard basis by

$$g(\sigma, \epsilon)e_i = \epsilon(i)e_{\sigma(i)}$$

where $\sigma \in S_n$ and ϵ ranges among the functions from $\{1, \ldots, n\}$ to $C(r)$.

The group $G(r, p, n)$ is the subgroup of $G(r, 1, n)$ consisting of all the $g(\sigma, \epsilon)$ such that the product $\epsilon(1)\epsilon(2)\cdots\epsilon(n)$ is a power of ζ^p. If $p < r$ the sets of reflecting hyperplanes of $G(r, p, n)$ and $G(r, 1, n)$ coincide, and their intersection lattice is the Dowling lattice $Q_n(\mathbb{Z}^r)$ (see [15]).

As important examples, we observe that, for $n \geq 2$, $G(1, 1, n) = S_n$ is the Weyl group of type A_{n-1}, while $G(2, 1, n)$ is the Weyl group of type B_n and, for $n \geq 4$, $G(2, 2, n)$ is the Weyl group of type D_n.

1.1. The interest of the models $Y_{G(r,p,n)}$

Wonderful models have been constructed by De Concini–Procesi in their seminal papers [7] and [8]. They play a relevant role in several fields: subspace and toric arrangements (see [10,22]), configuration spaces, box splines and index theory (see the exposition in [9]), tropical geometry (see for instance [21] and the survey [12]) and discrete geometry (see [18] for further references). We will recall in Sections 2.1 and 2.2 the construction of these models, including the definitions of nested sets and building sets, and their main properties. The importance of the models associated with reflection groups, i.e. with the hyperplane arrangements given by their reflecting hyperplanes, was at first pointed out by the example of type A: the minimal projective De Concini–Procesi model of type A_{n-1} is isomorphic to the moduli space $\overline{M}_{0,n+1}$ of $(n + 1)$-pointed stable curves of genus 0. This isomorphism carries on the cohomology of the models of type A_{n-1} an 'hidden' extended action of S_{n+1} that has been studied by several authors (see for instance [28,36,17]).

Also the other models $Y_{G(r,p,n)}$ appeared in the literature in several contexts. They are crucial objects in representation theory, since they provide natural geometric representations of $G(r, p, n)$. They were studied from this point of view by Henderson in [29], where recursive character formulas for the action of $G(r, 1, n)$ and $G(r, r, n)$ on their cohomology were described, as well as their specializations that give recursive formulas for the Betti numbers. We note here that the model $Y_{G(r,1,n)}$ is equal to $Y_{G(r,p,n)}$ if $p < r$, since the underlying reflection arrangement is the same. We recall that recursive formulas for the Betti numbers in the cases A_n, B_n and D_n have been obtained also in [38] and [23] (for the A_n case these formulas have been found in several other papers devoted to the moduli spaces approach, see for instance [32]).

We remark that in the case of a finite real reflection group G, one can construct a complex minimal model Y_G and also a minimal real compact model $\overline{Y}_G$: formulas for the action on the cohomology of $\overline{Y}_G$ in the A_n case appear in [36] while in [30] the cases of the other finite Coxeter groups are dealt with.

The combinatorial and discrete geometric interest of the real models $\overline{Y}_G$ comes from the observation that they can be obtained by glueing some nestohedra: for instance, the models $\overline{Y}_{G(1,1,n)}$ and $\overline{Y}_{G(2,1,n)}$ are obtained by glueing Stasheff's associahedra, while the models $\overline{Y}_{G(2,2,n)}$ are obtained by glueing graph associahedra of type D_n (in the sense of Carr and Devadoss, see [5]). There are also non minimal De Concini–Procesi models (see [27] for a classification), whose construction involves the glueing of permutohedra and other nestohedra (see [25]).

Finally we would like to mention that the minimal complex model Y_G, when G is an irreducible finite complex reflection group, plays a role in the theory of braid groups: for instance, the elements in the center of the pure braid group PB_G (resp. the braid group B_G) associated to G are easily described in terms of the geometry of Y_G (resp. Y_G/G), as well as the elements in the center of the parabolic subgroups of PB_G (resp. B_G, see [4]).

1.2. A combinatorial approach

In [3] a new exponential (non recursive) formula for the Betti numbers of the models $Y_{G(1,1,n)}$ has been found, using the following combinatorial approach. In [38] (see also [23]) a monomial basis of $H^*(Y_{G(1,1,n)})$ was described; the elements of this basis can be represented by graphs, that are some oriented rooted trees on n leaves, with exponents attached to the internal vertices. Now let us focus