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Connectivity for bridge-alterable graph classes

Colin McDiarmid

Department of Statistics, Oxford University, 24-29 St Giles', Oxford OX1 3LB, UK

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ABSTRACT

A collection $\mathcal{A}$ of graphs is called bridge-alterable if, for each graph G with a bridge e, G is in $\mathcal{A}$ if and only if $G-e$ is. For example the class $\mathcal{F}$ of forests is bridge-alterable. For a random forest F_n sampled uniformly from the set $\mathcal{F}_n$ of forests on vertex set $\{1, \ldots, n\}$, a classical result of Rényi (1959) shows that the probability that F_n is connected is $e^{-\frac{1}{2}+o(1)}$.

Recently Addario-Berry et al. (2012) and Kang and Panagiotou (2013) independently proved that, given a bridge-alterable class $\mathcal{A}$, for a random graph R_n sampled uniformly from the graphs in $\mathcal{A}$ on $\{1, \ldots, n\}$, the probability that R_n is connected is at least $e^{-\frac{1}{2}+o(1)}$. Here we give a more straightforward proof, and obtain a stronger non-asymptotic form of this result, which compares the probability to that for a random forest. We see that the probability that R_n is connected is at least the minimum over $\frac{2}{5}n < t \leq n$ of the probability that F_t is connected.

1. Introduction

A collection $\mathcal{A}$ of graphs is *bridge-addable* if for each graph G in $\mathcal{A}$ and pair of vertices u and v in different components, the graph $G+uv$ obtained by adding the edge (bridge) uv is also in $\mathcal{A}$; that is, if $\mathcal{A}$ is closed under adding bridges. This property was introduced in [9] (under the name 'weakly addable'). If also $\mathcal{A}$ is closed under deleting bridges we call $\mathcal{A}$ *bridge-alterable*. Thus $\mathcal{A}$ is bridge-alterable exactly when, for each graph G with a bridge e, G is in $\mathcal{A}$ if and only if $G-e$ is in $\mathcal{A}$. The class $\mathcal{F}$ of forests is bridge-alterable, as for example is the class of series–parallel graphs, the class of planar graphs, and indeed the class of graphs embeddable in any given surface. All natural examples of bridge-addable classes seem to satisfy the stronger condition of being bridge-alterable.

E-mail address: cmcd@stats.ox.ac.uk.

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Given a class $\mathcal{A}$ of graphs we let $\mathcal{A}_n$ denote the set of graphs in $\mathcal{A}$ on vertex set $[n] := \{1, \ldots, n\}$. Also, we use the notation $R_n \in_u \mathcal{A}$ to mean that R_n is a random graph sampled uniformly from $\mathcal{A}_n$ (where we assume implicitly that $\mathcal{A}_n$ is non-empty).

For a random forest $F_n \in_u \mathcal{F}$, a classical result of Rényi [13] from 1959 shows that, as $n \to \infty$

$$\mathbb{P}(F_n \text{ is connected}) = e^{-\frac{1}{2}+o(1)}. \tag{1}$$

In their investigations on random planar graphs, McDiarmid, Steger and Welsh [9] showed that, when $\mathcal{A}$ is bridge-addable, for $R_n \in_u \mathcal{A}$

$$\mathbb{P}(R_n \text{ is connected}) \geq e^{-1}. \tag{2}$$

It was observed by the same authors [10] in 2006 that the class of forests seems to be the 'least connected' bridge-addable class of graphs, and they made the following conjecture.

Conjecture 1.1. *When $\mathcal{A}$ is bridge-addable, for $R_n \in_u \mathcal{A}$*

$$\mathbb{P}(R_n \textit{ is connected}) \geq e^{-\frac{1}{2}+o(1)}.$$

This conjecture was then strengthened (see Conjecture 1.2 of [3], Conjecture 5.1 of [1], or Conjecture 6.2 of [8]) to the following non-asymptotic form.

Conjecture 1.2. *When $\mathcal{A}$ is bridge-addable, for $R_n \in_u \mathcal{A}$*

$$\mathbb{P}(R_n \textit{ is connected}) \geq \mathbb{P}(F_n \textit{ is connected}).$$

Early progress was made on Conjecture 1.1 by Balister, Bollobás and Gerke [2,3]; and recently Norin [12] made further progress, showing that $\mathbb{P}(R_n \text{ is connected}) \geq e^{-\frac{2}{3}+o(1)}$. Addario-Berry, McDiarmid and Reed [1] and Kang and Panagiotou [6] independently proved the following theorem, which establishes the special case of Conjecture 1.1 when $\mathcal{A}$ is bridge-alterable.

Theorem 1.3 *([1,6]). Let $\mathcal{A}$ be a bridge-alterable class of graphs, and let $R_n \in_u \mathcal{A}$. Then*

$$\mathbb{P}(R_n \textit{ is connected}) \geq e^{-\frac{1}{2}+o(1)}.$$

Here we give a reasonably short and straightforward proof of the following non-asymptotic form of this result, which together with (1) gives Theorem 1.3. This is a first step towards Conjecture 1.2, at least for a bridge-alterable class.

Theorem 1.4. *Let $\mathcal{A}$ be a bridge-alterable class of graphs, let n be a positive integer, let $R_n \in_u \mathcal{A}$, and let $F_t \in_u \mathcal{F}$ for $t = 1, 2, \ldots$. Let $\alpha = 0.4$. Then*

$$\mathbb{P}(R_n \textit{ is connected}) \geq \min_{\alpha n \leq t \leq n} \mathbb{P}(F_t \textit{ is connected}). \tag{3}$$

The value $\alpha = 0.4$ can be increased towards $\frac{1}{2}$: in the final section of the paper we improve it to $0.48n$, and discuss pushing it up further to $\frac{1}{2}$. Conjecture 1.2 says that we can push α up to 1.

Since this paper was (essentially) completed, the original Conjecture 1.1 (for bridge-addable rather than bridge-alterable classes) has been fully proved by Chapuy and Perarnau, see [4].

2. Proof of Theorem 1.4

We use two lemmas in the proof.

Lemma 2.1. *Let $\mathcal{A}$ be a bridge-alterable class of graphs, let n be a positive integer, let $R_n \in_u \mathcal{A}$, and let $F_t \in_u \mathcal{F}$ for $t = 1, 2, \ldots$. Then*

$$\mathbb{P}(R_n \textit{ is connected}) \geq \min_{t=1,\ldots,n} \max\left\{e^{-\frac{t}{n}}, \mathbb{P}(F_t \textit{ is connected})\right\}. \tag{4}$$