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Linear independence, a unifying approach to shadow theorems

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ABSTRACT

The intersection shadow theorem of Katona is an important tool in extremal set theory. The original proof is purely combinatorial. The aim of the present paper is to show how it is using linear independence latently.

1. Introduction

Let $[n] = \{1, 2, \ldots, n\}$ be the standard n-element set and let $\binom{[n]}{k}$ denote the collection of all its k-element subsets.

A family $\mathcal{F} \subset \binom{[n]}{k}$ is called *t-intersecting* if $|F \cap F'| \geqq t$ holds for all $F, F' \in \mathcal{F}$, $n \geqq k \geqq t > 0$.

For an integer s, $0 \leqq s \leqq k$, the *s-shadow*, $\Delta_s(\mathcal{F})$ is defined by

$$\Delta_s(\mathcal{F}) = \{G : \ |G| = s, \ \exists F \in \mathcal{F}, \ G \subset F\}.$$

Katona Intersecting Shadow Theorem ([7]). *If $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting, then*

$$\left|\Delta_s(\mathcal{F})\right| \geqq |\mathcal{F}| \times \binom{2k-t}{s} \Big/ \binom{2k-t}{k} \tag{1}$$

holds for all s, $k - t \leqq s \leqq k$.

Let us note that in the above range the factor of $|\mathcal{F}|$ on the right-hand side is at least one.

Choosing $\mathcal{F} = \binom{[2k-t]}{k}$ shows that (1) is best possible.

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Katona's proof was purely combinatorial. It relied on *shifting*, an operation on subsets and families of sets, invented by Erdős, Ko and Rado [2]. We are not giving here the rather technical definition but mention that it maintains the size of the subsets, the size of the family, the t-intersecting property and it does not increase the size of the shadow.

Repeated applications of shifting produce a family having the following property.

$$\text{For } 1 \leqq i < j \leqq n, \quad \text{if } j \in F \in \mathcal{F} \quad \text{and} \quad i \notin F \quad \text{then } (F - \{j\}) \cup \{i\} \in \mathcal{F} \text{ also.} \tag{2}$$

A family $\mathcal{F}$ satisfying (2) is called *shifted*.

The author proved the following

Claim 1 (*[3]*)**.** *If $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting and shifted, then for every $F \in \mathcal{F}$ there exists an $\ell = \ell(F)$, $0 \leqq \ell \leqq k - t$ such that*

$$\left|F \cap [t + 2\ell]\right| \geqq t + \ell \quad \textit{holds}. \tag{3}$$

Note that for a fixed ℓ one can define

$$\mathcal{A}_\ell(n, k, t) = \left\{A \in \binom{[n]}{k} : A \cap [t + 2\ell] \geqq t + \ell\right\}.$$

These are usually called the Frankl-families and they are t-intersecting. It was conjectured in [3] and proved by Ahlswede and Khachatrian [1] that

$$|\mathcal{F}| \leqq \max_{\ell} \left|\mathcal{A}_\ell(n, k, t)\right| \tag{4}$$

holds for every t-intersecting family $\mathcal{F} \subset \binom{[n]}{k}$, $n \geqq 2k - t$. Let us define $\mathcal{F}(n, k, t)$ as the family of all $F \in \binom{[n]}{k}$ satisfying (3) for *some* ℓ. Obviously, $\mathcal{F}(n, k, t)$ is the union of $\mathcal{A}_\ell(n, k, t)$ for $0 \leqq \ell \leqq k - t$. In view of Claim 1 we have

Claim 2. *If $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting and shifted, then $\mathcal{F} \subset \mathcal{F}(n, k, t)$ holds.*

Note that $\mathcal{F}(n, k, t)$ is no longer t-intersecting for $n > 2k - t$. However, the author proved that it still verifies (1).

Proposition 1 (*[4]*)**.** *If $\mathcal{F} \subset \mathcal{F}(n, k, t)$, then (1) holds.*

2. Inclusion matrices and statement of the results

Let $0 \leqq r < k \leqq n$ be integers and $\mathcal{F} \subset \binom{[n]}{k}$ a family of subsets. One defines the inclusion matrix $M(r, \mathcal{F})$ as a $(0 - 1)$-matrix whose rows are indexed by the subsets $G \in \binom{[n]}{r}$, and columns are indexed by the members F of $\mathcal{F}$. The general entry is 1 if $G \subset F$ and 0 otherwise.

The ordering of the rows is not essential but for convenience we use the *colex* order i.e., G precedes G' iff the maximal element of $G \setminus G'$ is smaller than that of $G' \setminus G$.

Note that the (column) vector $\overrightarrow{w}(F)$ is a vector of length $\binom{n}{r}$ having $\binom{k}{r}$ entries equal to 1, corresponding to the sets in $\binom{F}{r}$.

Füredi and the author proved the following.

Theorem (*[5]*)**.** *If the rank of $M(k - t, \mathcal{F})$ is $|\mathcal{F}|$, then $\mathcal{F}$ verifies (1).*