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# Edge-disjoint rainbow spanning trees in complete graphs

James M. Carraher [a], Stephen G. Hartke [b], Paul Horn [c]

[a] Department of Mathematics and Statistics, University of Nebraska at Kearney, United States

[b] Department of Mathematical and Statistical Sciences, University of Colorado Denver, United States

[c] Department of Mathematics, University of Denver, United States

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ABSTRACT

Let $G$ be an edge-colored copy of $K_n$, where each color appears on at most $n/2$ edges (the edge-coloring is not necessarily proper). A rainbow spanning tree is a spanning tree of $G$ where each edge has a different color. Brualdi and Hollingsworth (1996) conjectured that every properly edge-colored $K_n$ ($n \geq 6$ and even) using exactly $n-1$ colors has $n/2$ edge-disjoint rainbow spanning trees, and they proved there are at least two edge-disjoint rainbow spanning trees. Kaneko et al. (2003) strengthened the conjecture to include any proper edge-coloring of $K_n$, and they proved there are at least three edge-disjoint rainbow spanning trees. Akbari and Alipour (2007) showed that each $K_n$ that is edge-colored such that no color appears more than $n/2$ times contains at least two rainbow spanning trees.

We prove that if $n \geq 1{,}000{,}000$, then an edge-colored $K_n$, where each color appears on at most $n/2$ edges, contains at least $\lfloor n/(1000 \log n) \rfloor$ edge-disjoint rainbow spanning trees.



## 1. Introduction

Let $G$ be an edge-colored copy of $K_n$, where each color appears on at most $n/2$ edges (the edge-coloring is not necessarily proper). A rainbow spanning tree is a spanning tree of $G$ such that each edge has a different color. Brualdi and Hollingsworth [4] conjectured that every properly edge-colored $K_n$

*E-mail addresses:* carraherjm2@unk.edu (J.M. Carraher), stephen.hartke@ucdenver.edu (S.G. Hartke), paul.horn@du.edu (P. Horn).

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($n \geq 6$ and even) where each color class is a perfect matching has a decomposition of the edges of $K_n$ into $n/2$ edge-disjoint rainbow spanning trees. They proved there are at least two edge-disjoint rainbow spanning trees in such an edge-colored $K_n$. Kaneko, Kano, and Suzuki [13] strengthened the conjecture to say that for any proper edge-coloring of $K_n$ ($n \geq 6$) contains at least $\lfloor n/2 \rfloor$ edge-disjoint rainbow spanning trees, and they proved there are at least three edge-disjoint rainbow spanning trees. Akbari and Alipour [1] showed that each $K_n$ that is an edge-colored such that no color appears more than $n/2$ times contains at least two rainbow spanning trees.

Our main result is

**Theorem 1.** *Let $G$ be an edge-colored copy of $K_n$, where each color appears on at most $n/2$ edges and $n \geq 1{,}000{,}000$. The graph $G$ contains at least $\lfloor n/(1000 \log n) \rfloor$ edge-disjoint rainbow spanning trees.*

The strategy of the proof of Theorem 1 is to randomly construct $\lfloor n/(1000 \log n) \rfloor$ edge-disjoint subgraphs of $G$ such that with high probability each subgraph has a rainbow spanning tree. This result is the best known for the conjecture by Kaneko, Kano, and Suzuki. Horn [12] has shown that if the edge-coloring is a proper coloring where each color class is a perfect matching, then there are at least $\epsilon n$ rainbow spanning trees for some positive constant $\epsilon$, which is the best known result for the conjecture by Brualdi and Hollingsworth.

There have been many results in finding rainbow subgraphs in edge-colored graphs; Kano and Li [14] surveyed results and conjecture on monochromatic and rainbow (also called heterochromatic) subgraphs of an edge-colored graph. Related work includes Brualdi and Hollingsworth [5] finding rainbow spanning trees and forests in edge-colored complete bipartite graphs, and Constantine [8] showing that for certain values of $n$ there exists a proper coloring of $K_n$ such that the edges of $K_n$ decompose into isomorphic rainbow spanning trees.

The existence of rainbow cycles has also been studied. Albert, Frieze, and Reed [2] showed that for an edge-colored $K_n$ where each color appears at most $\lceil cn \rceil$ times then there is a rainbow hamiltonian cycle if $c < 1/64$. (Rue (see [11]) provided a correction to the constant.) Frieze and Krivelevich [11] proved that there exists a $c$ such that if each color appears at most $\lceil cn \rceil$ times, then there are rainbow cycles of all lengths.

This paper is organized as follows. Section 2 includes definitions and results used throughout the paper. Sections 3–5 contain lemmas describing properties of the random subgraphs we generate. The final section provides the proof of our main result.

## 2. Definitions

First we establish some notation that we will use throughout the paper. Let $G$ be a graph and $S \subseteq V(G)$. Let $G[S]$ denote the induced subgraph of $G$ on the vertex set $S$. Let $[S, \overline{S}]_G$ be the set of edges between $S$ and $\overline{S}$ in $G$. For natural numbers $q$ and $k$, $[q]$ represents the set $\{1, \dots, q\}$, and $\binom{[q]}{k}$ is the collection of all $k$-subsets of $[q]$. Throughout the paper the logarithm function used has base $e$. One inequality that we will use often is the union sum bound which states that for events $A_1, \dots, A_r$

$$\mathbb{P}\left[\bigcup_{i=1}^{r} A_i\right] \leq \sum_{i=1}^{r} \mathbb{P}\left[A_i\right].$$

Throughout the rest of the paper let $G$ be an edge-colored copy of $K_n$, where the set of edges of each color has size at most $n/2$, and $n \geq 1{,}000{,}000$. We assume $G$ is colored with $q$ colors, where $n-1 \leq q \leq \binom{n}{2}$. Let $C_j$ be the set of edges of color $j$ in $G$. Define $c_j = |C_j|$, and without loss of generality assume $c_1 \geq c_2 \geq \cdots \geq c_q$. Note that $1 \leq c_j \leq n/2$ for all $j$.

Let $t = \lfloor n/(C \log n) \rfloor$ where $C = 1000$. Note that we have not optimized the constant $C$, and it can be slightly improved at the cost of more calculation. Since $\frac{n}{C \log n} - 1 \leq t \leq \frac{n}{C \log n}$ we have

$$\frac{-1}{t} \leq \frac{-C \log n}{n} \quad \text{and} \quad \frac{C \log n}{n} \leq \frac{1}{t} \leq \left(\frac{n}{n - C \log n}\right) \frac{C \log n}{n}. \tag{$*$}$$

We will frequently use these bounds on $t$.