European Journal of Combinatorics 54 (2016) 35–50

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Locally planar graphs are 2-defective 4-paintable

Ming Han, Xuding Zhu

College of Mathematics, Physics and Information Engineering, Zhejiang Normal University, Jinhua 321004, China

ARTICLE INFO

Article history:
Received 22 December 2014
Accepted 7 December 2015
Available online 24 December 2015

ABSTRACT

The d-defective k-painting game on a graph G is played by two players: Lister and Painter. Initially, each vertex has k tokens and is uncoloured. In each round, Lister chooses a set M of uncoloured vertices and removes one token from each chosen vertex. Painter colours a subset X of M which induces a subgraph $G[X]$ of maximum degree at most d. Lister wins the game if at the end of some round, a vertex v has no more tokens left, and is uncoloured. Otherwise, at some round, all vertices are coloured and Painter wins. We say G is d-defective k-paintable if Painter has a winning strategy in this game. This paper proves that for each surface S, there is a constant w such that graphs embedded in S with edge-width at least w are 2-defective 4-paintable.

1. Introduction

A *d-defective colouring* of a graph G is a colouring of the vertices of G so that each colour class induces a subgraph of maximum degree at most d. Thus a 0-defective colouring of G is simply a proper colouring of G. Defective colouring of graphs was introduced by Cowen, Cowen and Woodall [1]. They proved that every outerplanar graph is 2-defective 2-colourable and every planar graph is 2-defective 3-colourable.

A *k-list assignment* is a mapping L which assigns to each vertex v a set $L(v)$ of k permissible colours. A *d-defective L-colouring* of G is a d-defective colouring c of G for which $c(v) \in L(v)$ for every $v \in V(G)$. A graph G is *d-defective k-choosable* if for any k-list assignment L of G, there exists a d-defective

E-mail address: xudingzhu@gmail.com (X. Zhu).

http://dx.doi.org/10.1016/j.ejc.2015.12.004
0195-6698/

L-colouring of G. Škrekovski [11] and Eaton and Hull [4] independently extended the above result to the list version and proved that every planar graph is 2-defective 3-choosable and every outerplanar graph is 2-defective 2-choosable. They both asked the question whether every planar graph is 1-defective 4-choosable. One decade later, Cushing and Kierstead [2] answered this question in the affirmative.

This paper studies the on-line version of list colouring of graphs, defined through a two person game.

The *d-defective k-painting game on G* is played by two players: Lister and Painter. Initially, each vertex v has k tokens and is uncoloured. In each round, Lister chooses a set M of uncoloured vertices and removes one token from each chosen vertex. Painter colours a subset X of M which induces a subgraph $G[X]$ of maximum degree at most d. Lister wins if at the end of some round, there is an uncoloured vertex with no more tokens left. Otherwise, at some round, all vertices are coloured and Painter wins. We say G is *d-defective k-paintable* if Painter has a winning strategy in this game. More generally, let $f : V(G) \to N$ be a mapping which assigns to each vertex v a positive integer $f(v)$. The *d-defective f-painting game on G* is the same as the d-defective k-painting game, except that at the beginning of the game, each vertex v has $f(v)$ tokens instead of k tokens. If Painter has a winning strategy for the d-defective f-painting game on G, then we say G is *d-defective f-paintable*. The mapping f is called a *token function*. For a vertex v of G, we shall denote by $\Theta(v)$ the set of neighbours of v that are coloured in the same round as v. Thus in the d-defective painting game, Painter's move is required to satisfy the constraint that for any vertex v, $|\Theta(v)| \leq d$. In each round, vertices in the set M chosen by Lister are called *marked vertices* in that round.

Assume L is a k-list assignment of G. If at round i, Lister chooses $M_i = \{v : i \in L(v), v \notin X_1 \cup \cdots \cup X_{i-1}\}$, where X_j is the set of vertices coloured by Painter in round j, and if Painter wins the game, then the colouring Painter constructed is a d-defective L-colouring of G. Therefore if G is d-defective k-paintable, then G is d-defective k-choosable. The converse is not true. A d-defective k-paintable graph is also called an *on-line d-defective k-choosable* graph. In the d-defective k-painting game, each vertex is also given a set of k permissible colours, and it needs to be coloured by one of the permissible colours. However, the list is given on-line and Painter needs to colour the vertices on-line, i.e., start colouring vertices before knowing the whole lists. Thus it is natural that some d-defective k-choosable graphs may not be d-defective k-paintable. The 0-defective painting of graphs was studied in [5–10,13]. It was shown in [14] that $\theta_{2,2,4}$ is 0-defective 2-choosable but not 0-defective 2-paintable.

Although every planar graph is 2-defective 3-choosable and 1-defective 4-choosable, it remains as open questions whether or not every planar graph is 2-defective 3-paintable, or 1-defective 4-paintable.

In this paper, we study the defective f-painting game on a more general class of graphs. Assume S is a surface and G is a graph embedded in S. The *edge-width* $\mathrm{ew}(G)$ of G is the length of a shortest cycle which is non-contractible in S. If the edge-width of G is large, then G is locally planar, and shares many common properties with planar graphs. If G has no non-contractible cycle, then G is planar and its edge-width is infinite.

It was proved by Thomassen [12] that for any surface S there exists a constant w such that every graph that can be embedded in S with edge-width at least w is 5-colourable. This result was strengthened by DeVos, Kawarabayashi and Mohar, who proved in [3] that for every surface S there exists a constant w such that every graph that can be embedded in S with edge-width at least w is 5-choosable. This result was further strengthened by Han and Zhu [5] who proved that for every surface S there exists a constant w such that every graph that can be embedded in S with edge-width at least w is 5-paintable.

This paper studies the defective painting game on locally planar graphs. We prove Theorem 1.1 below, which extends the above results about defective choosability in two aspects, namely from defective choosability to defective paintability, and from planar graphs to locally planar graphs. However, the conclusion combines the weaker parts of the two results above: instead of 1-defective 4-choosable and 2-defective 3-choosable, we prove locally planar graphs are 2-defective 4-paintable.

Theorem 1.1. *For every surface S there exists a constant w such that every graph that can be embedded in S with edge-width at least w is 2-defective 4-paintable.*